1 Monday, November 7: Synchrotron Radiation for Beginners

An accelerated electron emits electromagnetic radiation. The most effective way to accelerate an electron is to use electromagnetic forces. Since electrons have mass, they can also be accelerated gravitationally. However, electrons are seldom in freefall for long. In addition, gravitational accelerations are frequently small. An acceleration $g = 980 \, \mathrm{cm \, s^{-2}}$ makes an electron radiate with a power

$$P = \frac{2e^2 g^2}{3c^3} = 5.5 \times 10^{-45} \, \mathrm{erg \, s^{-1}} \ . \tag{1}$$

This amount of power would be produced if the electron passed a proton at a distance

$$b = \left(\frac{e^2}{m_e g} \right)^{1/2} = 510 \, \mathrm{cm} \ . \tag{2}$$

Thus, if a free electron is falling near the Earth's surface, if there's a proton within five meters, the electrostatic acceleration will be greater than the gravitational acceleration.[1]

An electron experiences an electromagnetic force if it passes a positively charged ion; the light it emits in this case is bremsstrahlung. In the limit that the relative velocity $\vec{v}$ of the electron and ion is small ($v \ll c$), the electron experiences a pure electric force directed toward the ion. In the case of relativistic bremsstrahlung, the electron experiences a mixture of electric and magnetic fields in its instantaneous rest frame as the ion zips past.

There exist circumstances in which the electron experiences a purely magnetic force. Suppose, in some inertial frame of reference, there exists a uniform magnetic flux density $\vec{B}$, with a negligibly small electric field strength $\vec{E}$. In the universe, it is not difficult to find magnetic fields, ranging in strength from the microgauss fields of intergalactic space to the teragauss fields near the most highly magnetized neutron stars. They aren't usually uniform in strength, but the constant $\vec{B}$ approximation is a useful place to start. If the velocity of the electron is small, then (as we've seen in Problem Set 2) the electron has a circular orbit in a plane perpendicular to $\vec{B}$, with

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[1]This is just another way of stating that gravity is a pathetically feeble force.

angular frequency

$$\omega_{\text{cyc}} = \frac{Be}{m_e c} = 1.8 \times 10^7 \, \text{s}^{-1} \left(\frac{B}{1 \, \text{gauss}} \right) . \tag{3}$$

The radiation produced in this non-relativistic case is called *cyclotron radiation*. In the limit $v \to c$, the cyclotron radiation is monochromatic, with frequency $\nu = \omega_{\text{cyc}}/(2\pi)$. Thus, the Earth's magnetic field, with $B \sim 0.5 \, \text{gauss}$, would produce radio waves with $\nu \sim 1 \, \text{MHz}$. Producing visible light by the cyclotron process would require a much higher magnetic flux density: $B \sim 2 \times 10^8 \, \text{gauss}$. The power produced by cyclotron radiation is

$$P_{\text{cyc}} = \frac{2e^2 \omega_{\text{cyc}}^4 r^2}{3c^3} = \frac{2}{3} r_0^2 c \left(\frac{v_{\text{cyc}}}{c} \right)^2 B^2 \tag{4}$$

$$= 1.6 \times 10^{-15} \, \text{erg} \, \text{s}^{-1} \, \beta_{\text{cyc}}^2 \left(\frac{B}{1 \, \text{gauss}} \right)^2 , \tag{5}$$

where $v_{\text{cyc}} = \beta_{\text{cyc}} c = \omega_{\text{cyc}} r$ is the orbital velocity of the electron in the magnetic field $\vec{B}$.

For a relativistic electron moving through a magnetic field, the emitted radiation becomes more complicated, and thus more interesting. As we noted just a week ago, in an inertial frame where a charged particle is at highly relativistic speeds, with a Lorentz factor

$$\gamma = (1 - v^2/c^2)^{-1/2} \gg 1 , \tag{6}$$

the radiation from the charged particle is strongly beamed in the particle's direction of motion. The angular width of the beamed radiation is $\sim 1/\gamma$. Not only is the direction of radiation altered in the relativistic limit, but so is the total power radiated and the spectrum of radiation.

To begin, consider a charged particle of mass m and charge q in a uniform magnetic field $\vec{B}$. There is no electric field: $\vec{E} = 0$. The velocity of the particle, $\vec{v} = \vec{\beta} c$, is not necessarily small compared to the speed of light. The equation of motion, in its correct relativistic form, is

$$\frac{d}{dt}(\gamma m \vec{v}) = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) = \frac{q}{c} \vec{v} \times \vec{B} . \tag{7}$$

The rate at which work is done on the charged particle is

$$\frac{d}{dt}(\gamma m c^2) = q \vec{v} \cdot (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) = 0 . \tag{8}$$

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The constancy of the charged particle's energy implies that $\gamma = $ constant. Thus, the velocity vector $\vec{v}$ is constant in length, although changing in direction, and the relativistic equation of motion can be written in the form

$$\frac{d\vec{v}}{dt} = \frac{q}{\gamma mc}\vec{v} \times \vec{B} \ . \tag{9}$$

Since the acceleration of the charged particle is proportional to the cross product of $\vec{v}$ and $\vec{B}$, it's convenient to separate $\vec{v}$ into a component parallel to $\vec{B}$ and a component perpendicular to $\vec{B}$. If we look at $\vec{v}_{\parallel}$, the velocity component parallel to $\vec{B}$, we find, from equation (9),

$$\frac{d\vec{v}_{\parallel}}{dt} = 0 \ , \tag{10}$$

and thus $v_{\parallel}$ is constant. Since $v = (v_{\parallel}^2 + v_{\perp}^2)^{1/2}$ is also constant, it follows that $v_{\perp}$ must also be constant. The equation of motion for $\vec{v}_{\perp}$,

$$\frac{d\vec{v}_{\perp}}{dt} = \frac{q}{\gamma mc}\vec{v}_{\perp} \times \vec{B} \ , \tag{11}$$

then represents motion in a circle with a constant acceleration and a constant angular frequency

$$\omega_B = \frac{qB}{\gamma mc} \ . \tag{12}$$

Note that the frequency is lower by a factor $1/\gamma$ than it would be in the non-relativistic limit.

The combination of constant velocity parallel to $\vec{B}$ and circular motion perpendicular to $\vec{B}$ means that the net motion of the charged particle is *helical* (Figure 1), with a *pitch angle* given by $\tan\alpha = v_{\perp}/v_{\parallel}$. As the charged particle winds along its helix, it emits radiation at a rate

$$P = \frac{2q^2}{3c^3}\gamma^4 a_{\perp}^2 \ , \tag{13}$$

where I am using the relativistically correct equation, taking into account that the acceleration is entirely perpendicular to the charged particle's motion. The magnitude of the acceleration is

$$a_{\perp} = \frac{v_{\perp}^2}{r} = v_{\perp}\omega_B = (\beta_{\perp}c)\left(\frac{qB}{\gamma mc}\right) \ . \tag{14}$$
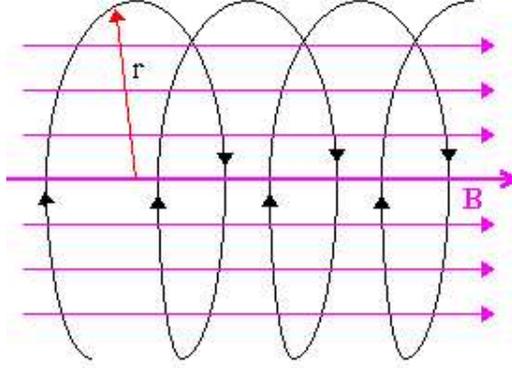
Figure 1: Helical motion of a charged particle in a magnetic field.

This leads to a radiated power of

$$P = \frac{2q^2}{3c^3}\gamma^4\beta_\perp^2 c^2 \frac{q^2 B^2}{\gamma^2 m^2 c^2} = \frac{2q^4}{3m^2 c^3}\gamma^2\beta_\perp^2 B^2 \ . \tag{15}$$

If the charged particle is an electron (and it usually is), we can write

$$P = \frac{2}{3}r_0^2 c\gamma^2\beta_\perp^2 B^2 \ , \tag{16}$$

larger by a factor of γ^2 than the non-relativistic result. The electrons in a magnetic field usually have a wide range of pitch angles α, so the power radiated by an individual electron,

$$P = \frac{2}{3}r_0^2 c\gamma^2\beta^2 \sin^2\alpha \, B^2 \ , \tag{17}$$

ranges from zero for electrons with $\alpha = 0$ (or $v_\perp = 0$) to a maximum for electrons with $\alpha = \pi/2$ (or $v_\parallel = 0$). If we assume that $\vec{v}$ has a probability distribution that is uniform over solid angle, then $\langle \sin^2\alpha \rangle = 2/3$, and

$$\langle P \rangle = \frac{4}{9}r_0^2 c\gamma^2\beta^2 B^2 \ . \tag{18}$$

The observed spectrum of synchrotron radiation is strongly affected by the fact that the radiation emitted by a relativistic electron is strongly beamed. Thus, an observer will only be able to see light from a highly relativistic electron during the part of its orbit when it's moving almost straight

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toward the observer. Thus, once for every turn of the helix, which occurs with period $t_B \equiv 2\pi/\omega_B$, the observer will detect a brief burst of radiation, with duration $\Delta t \ll t_B$. Since the power observed as a function of time is nearly a delta function, doing a Fourier transform tells us that the power observed as a function of angular frequency should be nearly uniform, up to a critical frequency $\omega_c \sim 1/\Delta t$.

The more detailed argument of Rybicki and Lightman tells us that the duration of a pulse seen by a distant observer should be

$$\Delta t \sim \frac{1}{\gamma^3} \frac{1}{\omega_B \sin \alpha} \ , \tag{19}$$

where α is the pitch angle.[2] Thus, we expect radiation from a synchrotron electron to have a more-or-less uniform spectrum up to a cutoff frequency

$$\omega_c \sim \frac{1}{\Delta t} \sim \gamma^3 \omega_B \sin \alpha \sim \gamma^2 \frac{eB}{m_e c} \sin \alpha \ . \tag{20}$$

There is a notable difference in the spectra of (highly non-relativistic) cyclotron radiation and (highly relativistic) synchrotron radiation. The cyclotron radiation is all emitted at the cyclotron frequency $\omega_{\mathrm{cyc}} = eB/m_e c$. The synchrotron radiation is emitted over a broad range of frequencies up to a limiting frequency $\omega_c \sim \gamma^2 \omega_{\mathrm{cyc}} \gg \omega_{\mathrm{cyc}}$. Thus, synchrotron radiation is a mechanism for extracting photons of high energy from a magnetic field of fixed B, and thus fixed cyclotron frequency ω_{cyc}.

2 Wednesday, November 9: Advanced Synchrotron Radiation

Suppose that an electron (charge $= -e$, mass $= m_e$) with velocity $\vec{v}$ is in a region of constant magnetic flux density $\vec{B}$. The velocity of the electron can also be expressed as the dimensionless velocity $\beta \equiv v/c$, as the Lorentz factor $\gamma \equiv (1 - \beta^2)^{-1/2}$, or as the electron energy $\epsilon = \gamma m_e c^2$. The electron

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[2]You might not have guessed γ^{-3} without doing the detailed analysis. A factor of γ^{-1} comes from geometrical considerations; the beam is only pointed toward you for a fraction $\sim 1/\gamma$ of the complete orbital period. A factor of γ^{-2} comes from the fact that the electron is moving toward the observer at highly relativistic speeds during the pulse, so the duration of the pulse is strongly blueshifted, by a factor $1 - v/c \sim 2/\gamma^2$.

will move on a helix whose axis lies along the $\vec{B}$ vector (see Figure 1). As the electron is accelerated, it emits electromagnetic radiation with power

$$P = \frac{2}{3}B^2 r_0^2 c\gamma^2\beta^2\sin^2\alpha = 2U_B\sigma_T c\gamma^2\beta^2\sin^2\alpha \ , \tag{21}$$

where $\sigma_T = (8\pi/3)r_0^2$ is the Thomson cross-section of the electron, and $U_B = B^2/(8\pi)$ is the energy density of the magnetic field. In the limit of an extremely relativistic electron, $\beta \approx 1$, and

$$P \approx 2U_B\sigma_T c\gamma^2\sin^2\alpha \ , \tag{22}$$

which becomes

$$\langle P\rangle \approx \frac{4}{3}U_B\sigma_T c\gamma^2 \tag{23}$$

when averaged over all angles, since $\langle\sin^2\alpha\rangle = 2/3$.

From the discussion of the previous lecture, we expect the spectral distribution of the power to be roughly constant at a value

$$\frac{dP}{d\omega} \sim \frac{P}{\omega_c} \ , \tag{24}$$

up to a critical frequency $\omega_c \sim \gamma^2\omega_{\mathrm{cyc}}\sin\alpha$, and drop off rapidly at higher frequencies. I was prowling through various textbooks trying to find a brief yet clear derivation of the actual shape of $dP/d\omega$. I was discouraged when Frank Shu stated (in *The Physics of Astrophysics, Volume I*), "The formal manipulations required for synchrotron theory can get formidable."[3] The full equations are laid out in Rybicki and Lightman. I will merely cite the results, then stride forward to apply them to something astrophysically interesting.

Physicists, by convention, have defined

$$\omega_c \equiv \frac{3}{2}\gamma^2\frac{eB}{m_e c}\sin\alpha = \frac{3}{2}\gamma^2\omega_{\mathrm{cyc}}\sin\alpha \ . \tag{25}$$

In terms of this critical frequency ω_c, the power spectrum of light from a highly relativistic synchrotron electron is

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{e^3 B\sin\alpha}{m_e c^2}F(\omega/\omega_c) \ . \tag{26}$$

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[3]Malcolm Longair (*High Energy Astrophysics, Volume II*), chimes in with "It is a very major undertaking to work out properly all the properties of synchrotron emission...".

The *synchrotron function* $F(x)$ is a dimensionless function which describes the shape of the synchrotron spectrum. The calculations in section 6.4 of Rybicki and Lightman reveal that

$$F(x) = x \int_x^\infty K_{5/3}(x')dx' , \qquad (27)$$

where $K_{5/3}$ is a modified Bessel function.[4] If you are a true Bessel function fanatic, you may want to prove that $F(x) \propto x^{1/3}$ when $x \ll 1$ (that is, when $\omega \ll \omega_c$), and that $F(x) \propto x^{1/2}e^{-x}$ when $x \gg 1$, giving the desired cutoff when $\omega \gg \omega_c$. A plot of $F(x)$ is shown in Figure 2. The synchrotron function
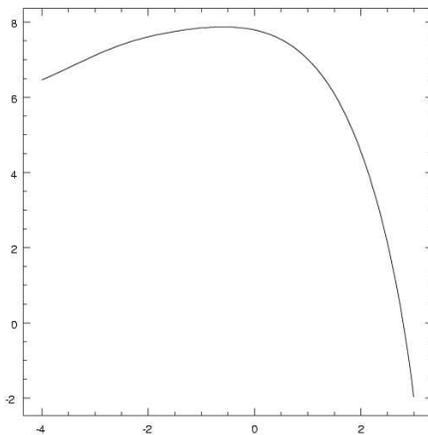
Figure 2: The synchrotron function $F(x)$, shown as $\log F$ versus $\log(\omega/\omega_c)$.

is skewed. The modal (most probable) value of x is $x_{\mathrm{mod}} \approx 0.29$; the mean value of x is $\langle x \rangle \approx 1.32$.

 If you had a population of electrons with the identical pitch angle α and Lorentz factor γ, you would expect the spectrum of light they emit to look like Figure 2, with

$$\frac{dP}{dV \, d\omega} = n_e \frac{dP}{d\omega} \propto \omega^{1/3} \qquad (28)$$

in the low frequency limit where $\omega \ll \omega_c$. The mean frequency of the emitted light would be

$$\langle \omega \rangle = 1.32\omega_c = 2.0\gamma^2\omega_{\mathrm{cyc}} \sin \alpha . \qquad (29)$$

However, when we look at the spectrum of actual synchrotron emitting sources, such as the Crab Nebula (Figure 3), we do not see a $\nu^{1/3}$ power

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[4]The factor of $\sqrt{3}/(2\pi)$ in equation (26) is then required for proper normalization.
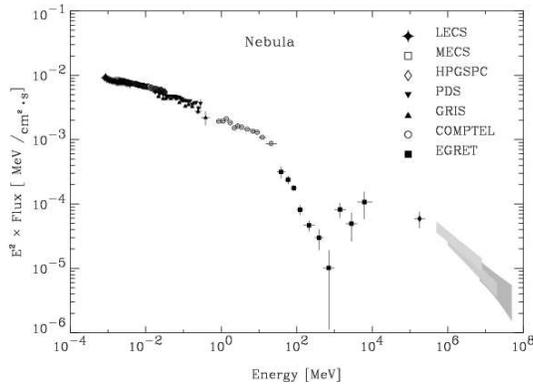
Figure 3: Spectrum of the Crab Nebula. (Note: what is plotted is $\propto \nu F_\nu$.)

law for I_ν. Frequently, the spectrum *is* reasonably well fit by a power law, $I_\nu \propto \nu^{-s}$, over a wide range of frequencies, but we usually find $s \neq -1/3$.

The fact that $s \neq -1/3$ can be easily explained: not all electrons have the same energy ϵ. In general, high energy electrons (large γ) are less common than low energy electrons (smaller γ). If you look, for instance, at the energy distribution for charged particles in cosmic rays, you find that a power law is a good fit:

$$n(\epsilon)d\epsilon \propto \epsilon^{-p}d\epsilon \ , \tag{30}$$

where $n(\epsilon)d\epsilon$ is the number density of electrons with energies in the range $\epsilon \rightarrow \epsilon + d\epsilon$. The fit is usual good over a wide range of energies: $\epsilon_1 < \epsilon < \epsilon_2$, with $\epsilon_2 \gg \epsilon_1$.[5] Since $\epsilon = \gamma m_e c^2$, a power-law distribution of energies implies a power-law distribution of Lorentz factors:

$$n(\gamma)d\gamma \propto \gamma^{-p}d\gamma \ , \tag{31}$$

for $\gamma_1 < \gamma\gamma_2$. To find the power radiated per unit volume per unit frequency, we need to compute

$$\frac{dP}{dV\,d\omega} = \int_{\gamma_1}^{\gamma_2} n(\gamma)\frac{dP}{d\omega}d\gamma \ . \tag{32}$$

Taking $dP/d\omega$ for a single electron from equation (26), and using the power-law distribution for $n(\gamma)$, we find that

$$\frac{dP}{dV\,d\omega} \propto \frac{e^3 B \sin\alpha}{m_e c^2} \int_{\gamma_1}^{\gamma_2} \gamma^{-p} F(x)d\gamma \ . \tag{33}$$

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[5]The observed cosmic ray spectrum has $p \approx 2.8$ from $\epsilon_1 \sim 10^{10}$ eV to $\epsilon_2 \sim 10^{20}$ eV.

where

$$x = \frac{\omega}{\omega_c} = \gamma^{-2}\frac{2\omega}{3\omega_{\text{cyc}}\sin\alpha} \ . \tag{34}$$

By changing the variable of integration from γ to $x \propto \gamma^{-2}$, we can write

$$\gamma^{-p}d\gamma = -\frac{1}{2}\left(\frac{2\omega}{3\omega_{\text{cyc}}\sin\alpha}\right)^{-(p-1)/2} x^{(p-3)/2}dx \ . \tag{35}$$

With this substitution in equation (33), we find that

$$\frac{dP}{dV\,d\omega} \ \propto \ \frac{e^3 B \sin\alpha}{m_e c^2}(\sin\alpha)^{(p-1)/2}\omega_{\text{cyc}}^{(p-1)/2}\omega^{-(p-1)/2}$$

$$\times \ \int_{x_2}^{x_1} x^{(p-3)/2}F(x)dx \ . \tag{36}$$

As long as $p > 1/3$, the integral over $F(x)$ converges, even in the limit $x_2 \to 0$ and $x_1 \to \infty$. It's just a dimensionless number of order unity.[6] By using the definition of the cyclotron frequency, $\omega_{\text{cyc}} = eB/(m_e c)$, we can write

$$\frac{dP}{dV\,d\omega} \ \propto \ \frac{e^3 B}{m_e c^2}(\sin\alpha)^{(p+1)/2}\left(\frac{eB}{m_e c}\right)^{(p-1)/2}\omega^{-(p-1)/2} \tag{37}$$

$$\propto \ \frac{e^2}{c}(\sin\alpha)^{(p+1)/2}\left(\frac{eB}{m_e c}\right)^{(p+1)/2}\omega^{-(p-1)/2} \ . \tag{38}$$

If we want, we can integrate over all pitch angles α, to leave the magnetic flux density B and the angular frequency ω as the only variables left in the problem. This yields the interesting result

$$\frac{dP}{dV\,d\omega} \propto B^{(p+1)/2}\omega^{-(p-1)/2} \ . \tag{39}$$

Note that the shape of the spectrum depends only on the parameter p, which describes the energy spectrum of the electrons. The shape of $F(x)$, which describes the radiation spectrum of an individual photon, only affects the normalization.[7]

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[6]It's actually something you can look up in integral tables, and involves the product of gamma functions.

[7]You find the same dependence of $dP/dV\,d\omega$ on ω if you substitute a delta function for $F(x)$.

We find, in conclusion, that if you have a power-law distribution of electron energies, $n(\epsilon) \propto \epsilon^{-p}$, you get out a power-law spectrum of synchrotron photons, $dP/dV\,d\omega \propto \omega^{-s}$, with $s = (p-1)/2$. If the spectrum of electron energies falls off very rapidly toward high energies, with $p \gg 1$, then the spectrum of synchrotron photons falls off less rapidly, with $s \approx p/2$. This is because $\langle \omega \rangle \propto \gamma^{-2}$; thus, a change that occurs over one e-folding of γ is stretched out over two e-foldings of ω. If the spectrum of electron energies falls off less precipitously, with $p \sim 1$, then the resulting synchrotron spectrum is nearly flat, with $s = (p-1)/2 \sim 0$. In many cases, it is found empirically that $p \approx 2.4$, leading to $s \approx 0.7$.

The continuum emission from the interstellar medium is commonly dominated by synchrotron emission at low frequency, where $I_\nu \propto \nu^{-0.7}$. The spectrum of the irregular galaxy M82, for instance, is shown in Figure 4. Synchrotron emission, which has a spectrum decreasing with frequency, is
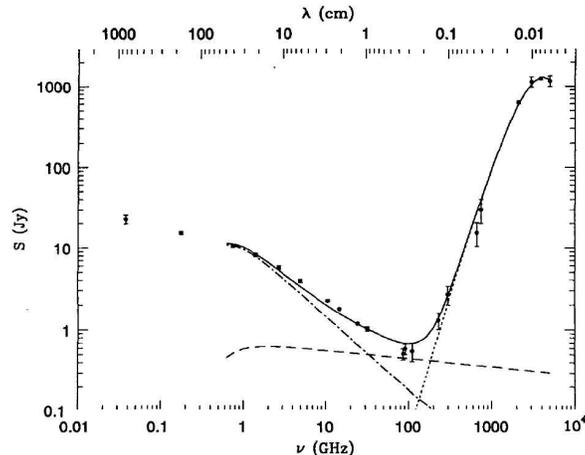
Figure 4: Contributions from synchrotron emission (dot-dashed), bremsstrahlung (dashed), and blackbody dust emission (dotted) to the spectrum of M82.

dominant in the low frequency regime ($\nu < 10,\mathrm{GHz}$, corresponding to $\lambda > 3\,\mathrm{cm}$). Bremsstrahlung, which has a nearly flat spectrum, dominates only at $\nu \sim 100\,\mathrm{GHz}$ (corresponding to $\lambda \sim 3\,\mathrm{mm}$. The blackbody emission from dust, which has $I_\nu \propto \nu^2$ in the Rayleigh-Taylor regime, dominates for $\nu > 300\,\mathrm{GHz}$, or $\lambda < 1\,\mathrm{mm}$.

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