Chapter 2

Viscosity, Heat Conduction, and Other Complications

The task at hand is to find expressions for the viscous stress tensor $\overleftrightarrow{\pi}$ and the conduction heat flux $\vec{F}$ in terms of the density, pressure, and bulk velocity of a gas. We will start with an ideal gas.

The distribution of random velocities for an ideal gas (otherwise known as an "perfect gas") is the Maxwellian distribution

$$f(\vec{w})d^3w = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mw^2}{2kT}\right) d^3w \ , \tag{2.1}$$

where k is the Boltzmann constant, m is the particle mass, and T is the kinetic temperature. The mean square random velocity is then $\langle w^2 \rangle = 3kT/m$. Using equation (1.31), we find that the pressure is given by the familiar ideal gas law

$$P = \frac{\rho}{m} kT \ . \tag{2.2}$$

For the Maxwellian distribution, $\langle w_i w_j \rangle = 0$ when $i \neq j$; hence, all elements of $\overleftrightarrow{\pi}$ vanish. Furthermore, $\langle w^2 w_i \rangle = 0$, so the conduction heat flux also vanishes. Ideal gases are delightfully simple: no viscosity, no heat conduction.

The conservation laws for an ideal gas take the simplified form

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{u} \tag{2.3}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \vec{\nabla} P + \vec{g} \tag{2.4}$$

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u}\ . \tag{2.5}$$

These three equations are known collectively as the **Euler equations**. In conjunction with the ideal gas law

$$\varepsilon = \frac{3P}{2\rho} = \frac{3kT}{2m}\ , \tag{2.6}$$

they describe the motions of an inviscid, ideal gas of point masses.[1] Combining the continuity and energy conservation equations, we see that for an ideal gas,

$$\frac{D\varepsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} = -P\frac{DV}{Dt}\ , \tag{2.7}$$

where $V \equiv 1/\rho$ is the specific volume of the gas. The change in internal energy of a ideal gas is equal to the PdV work that is done on the gas.

The first law of thermodynamics states that

$$Tds = d\varepsilon + PdV\ , \tag{2.8}$$

where T and P are the temperature and pressure of the gas, s is the specific entropy, ε is the specific internal energy, and V is the specific volume. Since, for an ideal gas,

$$\frac{D\varepsilon}{Dt} = -P\frac{DV}{Dt}\ , \tag{2.9}$$

it is necessary that

$$T\frac{Ds}{Dt} = 0\ . \tag{2.10}$$

An ideal gas, in the absence of heat sources and sinks, undergoes only **adiabatic** processes. (That is, its entropy remains constant.)

If a gas is compelled to have a constant volume (by being enclosed in a rigid box, for instance), the first law of thermodynamics reduces to $dq = d\varepsilon$, where dq is the heat added. The **specific heat** at constant volume is then

$$c_V \equiv \left(\frac{\partial q}{\partial T}\right)_V = \left(\frac{\partial \varepsilon}{\partial T}\right)_V\ . \tag{2.11}$$

However, for an ideal gas, $\varepsilon(V,T) = \varepsilon(T)$. Thus, we may write, quite generally,

$$d\varepsilon = c_V dT \tag{2.12}$$

[1]Inviscid = having no viscosity.

and

$$dq = c_V dT + PdV \ . \tag{2.13}$$

Now consider gas that is kept at a constant pressure. The ideal gas law tells us that

$$PdV = \frac{k}{m} dT \tag{2.14}$$

when pressure is kept constant, and hence that

$$c_P \equiv \left(\frac{\partial q}{\partial T} \right)_P = c_V + \frac{k}{m} \ . \tag{2.15}$$

Note that $c_P > c_V$; when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

The **adiabatic index** of a gas is defined as $\gamma \equiv c_P/c_V$. For an ideal gas of point particles, $\varepsilon = (3kT)/(2m)$, $c_V = (3k)/(2m)$, $c_P = (5k)/(2m)$ and $\gamma = 5/3$. The adiabatic index for a gas of diatomic molecules is $\gamma = 7/5$. The adiabatic index is a function of the number of degrees of freedom of the particles; diatomic molecules have rotational degrees of freedom that are not present for point masses. The internal energy ε, if the gas particles are not spherical atoms, also includes rotational energy in addition to the translational energy. In general, the internal energy is

$$\varepsilon = \frac{1}{\gamma - 1} \frac{kT}{m} = \frac{1}{\gamma - 1} \frac{P}{\rho} \ . \tag{2.16}$$

Now consider an adiabatic process, for which the first law of thermodynamics may be written

$$d\varepsilon + PdV = 0 \tag{2.17}$$

$$c_V dT - \frac{P}{\rho^2} d\rho = 0 \ . \tag{2.18}$$

For an ideal gas,

$$dT = \frac{mP}{k\rho} \left(\frac{dP}{P} - \frac{d\rho}{\rho} \right) \ . \tag{2.19}$$

Combining equations (2.18) and (2.19),

$$c_V \frac{mP}{k\rho} \left[\frac{dP}{P} - \left(\frac{c_V + k/m}{c_V} \right) \frac{d\rho}{\rho} \right] = 0 \ . \tag{2.20}$$

Thus, $dP/P = \gamma(d\rho/\rho)$, and

$$P(\rho) = P_0(\rho/\rho_0)^\gamma \ . \tag{2.21}$$

A gas that has an equation of state in this form ($P \propto \rho^\gamma$) is known as a **polytrope**.

An ideal gas has no viscosity. However, real gases aren't quite ideal, and in many astrophysical applications (such as accretion disks) viscosity is important. To handle viscosity in a relatively simple manner, we need to express the tensor $\overleftrightarrow{\pi}$ in terms of the bulk velocity $\vec{u}$. First, viscous frictional forces will occur when two fluid elements move relative to each other. Hence, $\overleftrightarrow{\pi}$ must depend on the spatial derivatives of the velocity, $\partial u_i/\partial x_j$. Second, viscous forces must disappear when the fluid is at rest or is in uniform translational or rotational motion. Third, for small velocity gradients, the elements of $\overleftrightarrow{\pi}$ will be *linearly proportional* to the velocity gradient. (Fluids for which $\pi_{ij} \propto \partial u_i/\partial x_j$ are known as linear fluids, or Newtonian fluids.)

The most general viscous stress tensor that satisfies these three requirements is

$$\pi_{ij} = \mu D_{ij} + \beta \vec{\nabla} \cdot \vec{u} \delta_{ij} \ , \tag{2.22}$$

where μ is the **coefficient of shear viscosity**, β is the **coefficient of bulk viscosity**, and

$$D_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\vec{\nabla} \cdot \vec{u}\delta_{ij} \ . \tag{2.23}$$

The **deformation tensor** D_{ij} vanishes in the case of uniform expansion or contraction. Thus, the shear viscosity term represents pure shear, with no change in volume, while the bulk viscosity term represents pure expansion or contraction.

In the cgs system, the coefficients of viscosity μ and β are measured in 'poises', where 1 poise equals 1 g/cm/sec.[2] At room temperature, the coefficient of shear viscosity for air is $\mu \sim 2 \times 10^{-4}$ poise; for water, $\mu \sim 10^{-2}$ poise.

The coefficient of shear viscosity can be approximately calculated in a fairly simple manner. A gas has a number density n of particles, each with a mass m. The gas has a temperature T, and hence a thermal velocity $v_t = (kT/m)^{1/2}$. In addition to the random thermal velocity of the particles,

[2] The 'poise' is named after the French physiologist Jean Poiseuille; as a consequence, it is pronounced in the French manner ('pwahz') rather than in the English manner ('poyz').

there is also a bulk velocity u_x in the x direction, where u_x is a function of y. (In other words, there exists a shear, $\partial u_x/\partial y \neq 0$.)

Now consider the plane $y = y_0$. Because of the random velocities of the particles, there is a flux of particles moving *downward* through the plane, with a magnitude per unit area of $\sim nv_t/2$. There is a flux of equal magnitude *upward* through the plane. Thus, there is no net flux of mass density through the plane. However, if $\partial u_x/\partial y > 0$, the downward flux of particles will have a greater momentum in the x direction than will the upward flux of particles. The net result: a transfer of momentum opposite to the direction of the velocity gradient $\partial u_x/\partial y$. The particles that cross the plane from above will have had their last collision at a distance $\sim \lambda$ above the plane. Thus, the downward flux of angular momentum will be

$$P_{yx} \sim \frac{nv_t}{2} m \left[u_x(y_0) + \lambda \frac{\partial u_x}{\partial y} \right] . \tag{2.24}$$

Similarly, the upward flux of angular momentum will be

$$P_{yx} \sim \frac{nv_t}{2} m \left[u_x(y_0) - \lambda \frac{\partial u_x}{\partial y} \right] . \tag{2.25}$$

The net momentum flux is then

$$P_{yx} \sim -nv_t m \lambda \frac{\partial u_x}{\partial y} , \tag{2.26}$$

which tells us that the coefficient of shear viscosity is

$$\mu \sim nv_t m \lambda \sim (mkT)^{1/2}/\sigma . \tag{2.27}$$

where σ is the cross sectional area of the gas particles. To lowest order, the coefficient of bulk viscosity, β, is equal to zero. Note that μ is independent of the density of the gas. Note also that the viscosity of an ideal gas increases with temperature, in contrast to the behavior of liquids, in which viscosity generally decreases with temperature. For neutral atomic hydrogen, the coefficient of shear viscosity (including relevant factors of π) is

$$\mu = 6 \times 10^{-3} \,\text{poise} \left(\frac{T}{10^4 \,\text{K}} \right)^{1/2} . \tag{2.28}$$

A dimensionless number that is much beloved of fluid dynamicists is the **Reynolds number**,

$$\mathrm{Re} \equiv \frac{\rho u L}{\mu} \sim \left(\frac{u}{v_t}\right)\left(\frac{L}{\lambda}\right) \ , \tag{2.29}$$

where L is the typical length scale of the system we're looking at. The Reynolds number is the ratio of the inertial forces ($\sim \rho u^2/L$) to the viscous forces ($\sim \mu u/L^2$). Thus, when $\mathrm{Re} \gg 1$, the viscous forces are negligible. As the scale of interest L becomes smaller and smaller, there is some length scale on which viscosity become important. The value of the Reynolds number determines the transition between *laminar* flow (small Re) and *turbulent* flow (large Re). At scales $L \gtrsim \mu/(\rho u)$, the flow will be turbulent; at smaller scales, viscosity will prevent turbulence from developing.

What about the conductive heat flux? It is found empirically that heat flows from hot regions to cold regions, with the flux proportional to the temperature gradient. Mathematically, this is expressed by Fourier's Law:

$$\vec{F} = -K\vec{\nabla} T \ , \tag{2.30}$$

where K is the **coefficient of thermal conductivity**.

For a neutral gas,

$$K = \frac{5}{2} c_V \mu \sim \frac{k}{\sigma}\left(\frac{kT}{m}\right)^{1/2} \ . \tag{2.31}$$

Thus, for a gas of neutral atomic hydrogen,

$$K = 2 \times 10^6 \,\mathrm{erg\,cm^{-1}\,s^{-1}\,K^{-1}} \left(\frac{T}{10^4\,\mathrm{K}}\right)^{1/2} \ . \tag{2.32}$$

Summary of Results So Far

The conservation equations are

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u} \tag{2.33}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi} + \vec{g} \tag{2.34}$$

$$\frac{D\varepsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u} - \frac{1}{\rho}\vec{\nabla}\cdot\vec{F} + \frac{1}{\rho}\Psi + \frac{1}{\rho}(\Gamma - \Lambda) \ , \tag{2.35}$$

in conjunction with the equation of state for an ideal monatomic gas

$$\varepsilon = \frac{3P}{2\rho} = \frac{3kT}{2m} \ . \tag{2.36}$$

In the simplest approximation, we ignore the viscosity and heat conduction by setting $\overleftrightarrow{\pi}$, $\vec{F}$, and Ψ equal to zero. When this approximation is made, the conservation equations are known as the Euler equations.

When viscosity and heat conduction are not negligible, we make the approximations

$$\pi_{ij} = \mu D_{ij} + \beta \vec{\nabla} \cdot \vec{u} \delta_{ij} \tag{2.37}$$

and

$$\vec{F} = -K \vec{\nabla} T \ . \tag{2.38}$$

In this approximation, the viscous force per unit volume is

$$\vec{\nabla} \cdot \overleftrightarrow{\pi} = \mu \nabla^2 \vec{u} + (\beta + \mu/3) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) \ . \tag{2.39}$$

The rate of viscous energy dissipation is equal to

$$\Psi = \frac{\mu}{2} | \overleftrightarrow{D} |^2 + \beta (\vec{\nabla} \cdot \vec{u})^2 \ , \tag{2.40}$$

where the square of the scalar norm of the deformation tensor is

$$| \overleftrightarrow{D} |^2 = \sum_{i,j} D_{ij} D_{ij} \ . \tag{2.41}$$

Chapter 3

Introduction to Sound & Shocks

Suppose we are in an inviscid, non-heat-conducting, nonradiative medium. We will first consider a case with plane parallel symmetry (all the properties of the gas depend only on x and t). The mass continuity equation is then

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \ . \tag{3.1}$$

The momentum equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g \ . \tag{3.2}$$

If the medium is self-gravitating, the acceleration g is given by Poisson's equation:

$$\frac{\partial g}{\partial x} = -4\pi G\rho \ . \tag{3.3}$$

A uniform static medium, with $\rho = \rho_0$, $u = 0$, and $P = P_0$, will satisfy the equations of continuity and of motion if we perpetuate the **Jeans swindle**. In a homogeneous, isotropic universe, the gravitational acceleration must be $\vec{g}_0 = 0$, by symmetry. However, Poisson's equation will only yield $g_0 = 0$ if $\rho_0 = 0$. From the Newtonian point of view, in other words, an infinite, static, matter-filled universe cannot exist. The Jeans swindle deals with this difficulty by ignoring it. Let us, like Jeans, assume that $g_0 = 0$ for the uniform medium.

Now, let us introduce small perturbations to the system, so that

$$\rho = \rho_0 + \rho_1(x,t) \quad (3.4)$$
$$u = u_1(x,t) \quad (3.5)$$
$$P = P_0 + P_1(x,t) \quad (3.6)$$
$$g = g_1(x,t) \ . \quad (3.7)$$

where $|\rho_1|/\rho_0 \ll 1$ and $|P_1|/P_0 \ll 1$. The linearized perturbation equations are then

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0 \quad (3.8)$$
$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{dP}{d\rho}|_0 \frac{\partial \rho_1}{\partial x} = g_1 \rho_0 \quad (3.9)$$
$$\frac{\partial g_1}{\partial x} = -4\pi G \rho_1 \ . \quad (3.10)$$

In writing the equation of motion, I have made the implicit assumption that $P = P(\rho)$. Taking the time derivative of equation (3.8), and subtracting the spatial derivative of equation (3.9), we find

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{dP}{d\rho}|_0 \frac{\partial^2 \rho_1}{\partial x^2} = 4\pi G \rho_0 \rho_1 \ . \quad (3.11)$$

If the self-gravitation term on the right hand side is small enough to be ignored, this is just a wave equation with a propagation speed

$$a = \left(\frac{dP}{d\rho} \right)^{1/2} \ . \quad (3.12)$$

The density and pressure perturbations that propagate through the medium with velocity $\pm a$ are **sound waves**. For a polytrope with adiabatic index γ, the sound speed at density ρ_0 and pressure P_0 is

$$a_0 = \left(\frac{\gamma P_0}{\rho_0} \right)^{1/2} = \left(\frac{\gamma k}{m} T \right)^{1/2} \ . \quad (3.13)$$

For a neutral atomic gas, the sound speed is $a = 0.12\,\mathrm{km\,s^{-1}}\,\mu_a^{-1/2}(T/1\,\mathrm{K})^{1/2}$, where μ_a is the mean atomic weight in units of the proton mass.

When is the self-gravity of the medium negligible? Consider a sound wave of the form $\rho_1(x,t) \propto \exp[i(\omega t - kx)]$. In that case, equation (3.11), including the self-gravity term on the right hand side, reduces to the dispersion relation

$$\omega^2 = k^2 a_0^2 - 4\pi G\rho_0 \ . \tag{3.14}$$

Thus, ω is real for wavenumbers $k > k_J$, where

$$k_J \equiv \sqrt{4\pi G\rho_0}/a_0 \ . \tag{3.15}$$

For $k < k_J$, the frequency ω is imaginary, and the perturbations grow exponentially due to their self-gravity.

The **Jeans length** for a neutral atomic gas is

$$\lambda_J \equiv \frac{2\pi}{k_J} \sim 20\,\mathrm{pc}\ \mu_a^{-1} \left(\frac{T}{1\,\mathrm{K}}\right)^{1/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \tag{3.16}$$

and the **Jeans mass** is

$$M_J \equiv \frac{\pi}{6}\lambda_J^3\rho_0 \sim 100\,\mathrm{M}_\odot\ \mu_a^{-2} \left(\frac{T}{1\,\mathrm{K}}\right)^{3/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \ . \tag{3.17}$$

Sound waves with a wavelength longer than λ_J will collapse gravitationally. On the other end of the size spectrum, sound waves with a wavelength shorter than the mean free path $\lambda = 1/(n\sigma)$ cannot be created. Even in relatively dense molecular clouds, $\lambda \sim 0.2\,\mathrm{AU}$; a sound with this wavelength will have a frequency of $\sim 10^{-8}\,\mathrm{Hz}$. From now on, we will deal with sound whose wavelength lies between the mean free path λ and the Jeans length λ_J, so that we can ignore the effects of discreteness and self-gravitation. But what about the effects of viscosity and heat conduction?

The linearized 1-d continuity equation, as always, is

$$\frac{\partial\rho_1}{\partial t} = -\rho_0\frac{\partial u_1}{\partial x} \ . \tag{3.18}$$

The momentum equation, including viscosity, is

$$\rho_0\frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} + \mu'\frac{\partial^2 u_1}{\partial x^2} \ , \tag{3.19}$$

where the effective coefficient of viscosity in the one-dimensional case is $\mu' \equiv 4\mu/3 + \beta$. The energy equation, including heat conduction, is

$$\rho_0\frac{\partial \varepsilon_1}{\partial t} = -P_0\frac{\partial u_1}{\partial x} + K\frac{\partial^2 T_1}{\partial x^2} \ . \tag{3.20}$$

The temperature perturbation, in terms of the density and pressure, is

$$T_1 = \frac{m}{k\rho_0}[P_1 - \frac{P_0}{\rho_0}\rho_1] \ . \tag{3.21}$$

The perturbation to the specific internal energy is

$$\varepsilon_1 = \frac{1}{(\gamma - 1)\rho_0}[P_1 - \frac{P_0}{\rho_0}\rho_1] \ . \tag{3.22}$$

Using these relations in conjunction with the continuity equation, the energy equation takes the form

$$\frac{\partial}{\partial t}(P_1 - a_0^2\rho_1) = \gamma\chi\frac{\partial^2}{\partial x^2}(P_1 - a_0^2\rho_1/\gamma) \ , \tag{3.23}$$

where $\chi = K/(\rho_0 c_P)$.

If the perturbations are sine waves, of the form

$$\begin{aligned} \rho_1 &= R\exp[i(\omega t - kx)] && (3.24)\\ P_1 &= P\exp[i(\omega t - kx)] && (3.25)\\ u_1 &= U\exp[i(\omega t - kx)] \ , && (3.26) \end{aligned}$$

then the conservation equations yield the relations

$$\begin{aligned} i\omega R - i\rho_0 kU &= 0 && (3.27)\\ -ikP + [i\omega\rho_0 + \mu' k^2]U &= 0 && (3.28)\\ (i\omega + \gamma k^2\chi)P - (i\omega + k^2\chi)a_0^2 R &= 0 \ . && (3.29) \end{aligned}$$

This set of equations yields the dispersion relation

$$\omega^2 = \frac{1 - ik^2\chi/\omega}{1 - i\gamma k^2\chi/\omega}a_0^2 k^2 + i\mu' k^2\omega/\rho_0 \ . \tag{3.30}$$

In the absence of viscosity and heat conduction, the dispersion relation is $\omega^2 = a_0^2k^2$. Suppose, however, that we add a small μ and K, so that the wavenumber is now $k = \omega/a_0 + k_1$, with $|k_1| \ll \omega/a_0$. The perturbation to the wavenumber is then

$$k_1 = -i\frac{\omega^2}{2a_0^3\rho_0}[\mu' + (\gamma - 1)\rho_0\chi] \ . \tag{3.31}$$

The density perturbation is now

$$\rho_1 = R \exp[i(\omega t - \omega x/a_0)] \exp(-x/L_1) \ , \tag{3.32}$$

with the attenuation length

$$L_1 = \frac{2a_0^3\rho_0}{\omega^2}[\mu' + (\gamma - 1)\rho_0\chi]^{-1} \ . \tag{3.33}$$

The viscosity and heat conduction damp the sound waves by converting the sound energy into random kinetic energy.

If we use the sound speed $a^2 \sim kT/m$ and viscosity $\mu \sim (mkT)^{1/2}/\sigma$ for an atomic gas, we find that the attenuation length is

$$L_1 \sim \frac{a_0^2}{\omega^2} n\sigma \sim \frac{\Lambda^2}{\lambda} \ , \tag{3.34}$$

where Λ is the wavelength of the propagating sound and λ is the mean free path of the gas. For sound to propagate, we require $\Lambda \gg \lambda$, and hence $L_1 \gg \Lambda$; the sound is not attenuated significantly until it has traveled for many wavelengths. In air at room temperature, $L_1 \sim 600\,\mathrm{km}(\omega/1000\,\mathrm{Hz})^{-2}$. In the interstellar medium (ISM), $L_1 \sim 700\,\mathrm{AU}(\omega/10^{-10}\,\mathrm{Hz})^{-2}$.

So far, we have been assuming that the sound waves are of infinitesimal magnitude, with $|\rho_1|/\rho_0 \ll 1$, $|P_1|/P_0 \ll 1$, and $u_1 \ll a_0$. As a consequence, we have assumed that the sound speed in the medium has the uniform value $a = a_0$. However, the sound speed is a function of density; for a polytrope, $a \propto \rho^{(\gamma-1)/2}$. As a consequence, crests of sound waves will travel more rapidly than troughs of sound waves, as illustrated in Figure 3.1. Although a wave on the surface of water can be triple-valued, as shown at time 3 in Figure 3.1, creating a "breaker", this is forbidden for a sound wave. Sound waves, therefore, will steepen until a **shock** forms. A shock front is a surface that marks a sudden jump in the density, pressure, and velocity of a gas. The shock front is supersonic – that is, it propagates at a velocity faster than the sound speed in the unshocked medium.

Shocks are ubiquitous in the ISM. Whenever the bulk velocity u is larger than the sound speed, you are likely to form shocks. For instance, there are shocks associated with:

- cloud - cloud collisions ($u \sim 10\,\mathrm{km\,s^{-1}}$),
- galaxy - galaxy collisions ($u \sim 300\,\mathrm{km\,s^{-1}}$),

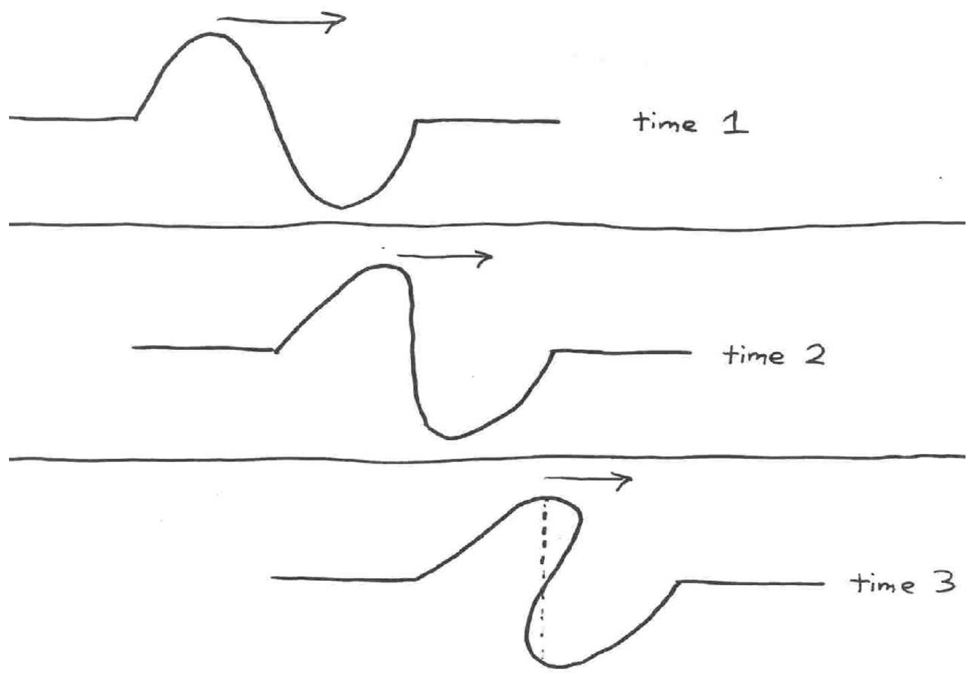

Figure 3.1: The steepening of a sound wave as a crest overtakes a trough.

- stellar winds encountering the ambient ISM ($u \sim 3000\,\mathrm{km\,s^{-1}}$),
- jets from radio galaxies encountering the ambient intergalactic medium ($u \sim 5000\,\mathrm{km\,s^{-1}}$),
- supernova ejecta encountering the ambient ISM ($u \sim 2 \times 10^4\,\mathrm{km\,s^{-1}}$), and
- gas accreting onto neutron stars ($u \sim 10^5\,\mathrm{km\,s^{-1}}$).

Shocks play an important role role in determining the structure of the ISM. For instance, they heat the ISM. Moreover, in the spiral arms of galaxies, shocks compress the gas, which can trigger star formation.

Let us see how shocks behave once they are created through the steepening of a sound wave. Consider, to begin with, a simple plane parallel shock, as shown in Figure 3.2. The math is easiest if we place ourselves in a frame of reference that is comoving with the shock front. Let u_1 be the bulk velocity of the unshocked gas (upstream of the shock) in this frame of reference, and let ρ_1, P_1, and a_1 be the density, pressure, and sound speed of the unshocked gas, which is assumed to be uniform. The bulk velocity, density, pressure, and sound speed immediately downstream from the shock are u_2, ρ_2, P_2, and a_2. The transition layer between the unshocked gas and the postshock gas is very thin. For a neutral atomic gas, $\Delta x \sim \lambda$, the mean free path length. For most purposes, it is adequate to regard the shock transition layer as an infinitesimally thin surface. For a steady-state shock, the conservation equations have the form

$$\frac{d}{dx}(\rho u) = 0 \tag{3.35}$$

$$\frac{d}{dx}(\rho u^2 + P) = 0 \tag{3.36}$$

$$\frac{d}{dx}(\rho[u^2/2 + \varepsilon] + P) = 0\ . \tag{3.37}$$

The gas properties immediately before and after being shocked are consequently linked by the **Rankine-Hugoniot jump conditions**:

$$\rho_1 u_1 = \rho_2 u_2 \tag{3.38}$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{3.39}$$

$$\frac{1}{2}u_1^2 + \varepsilon_1 + P_1/\rho_1 = \frac{1}{2}u_2^2 + \varepsilon_2 + P_2/\rho_2\ . \tag{3.40}$$

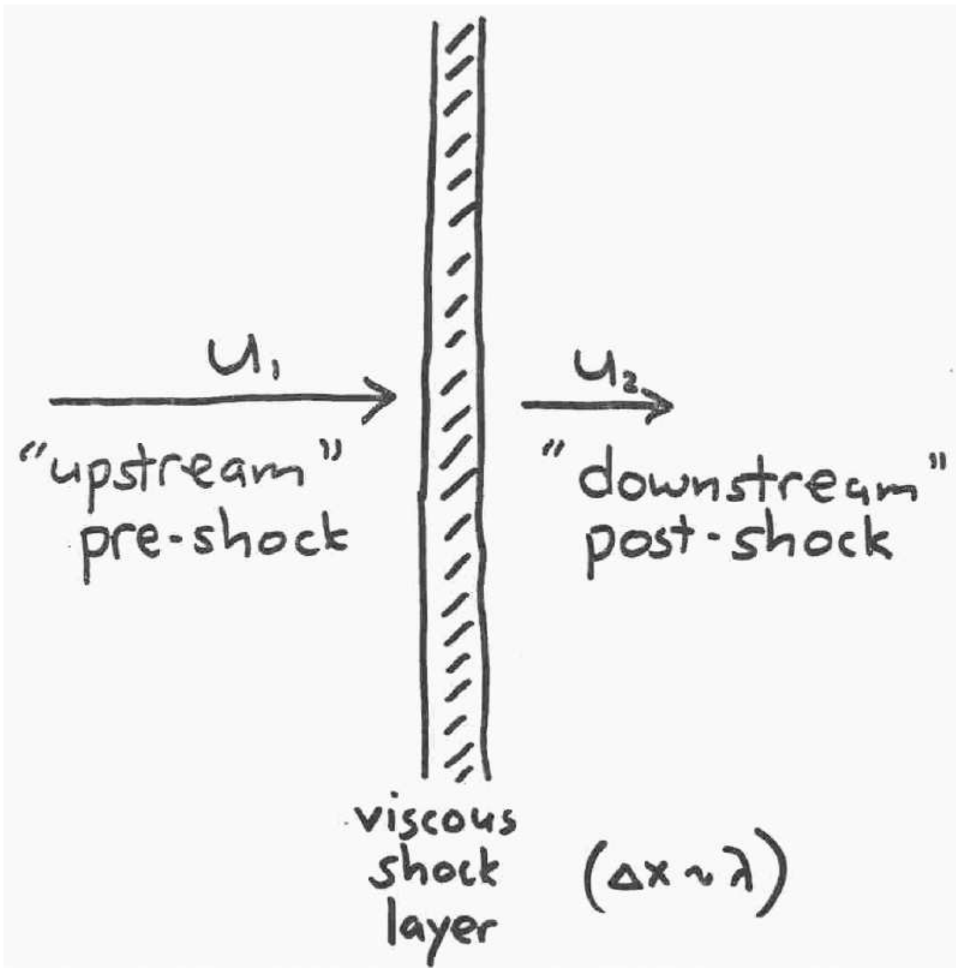

Figure 3.2: The geometry of a plane parallel shock; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

For a polytropic gas, the last of the Rankine-Hugoniot conditions may be rewritten in the form

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} \ . \tag{3.41}$$

I have made the implicit assumption that γ is the same for the preshock and postshock gas.

The Rankine-Hugoniot conditions are just the conservation equations in a new guise. A dimensionless number that is often cited in the context of shocks is the **Mach number**,

$$M_1 \equiv u_1/a_1 = \left(\frac{\rho_1 u_1^2}{\gamma P_1}\right)^{1/2} \ . \tag{3.42}$$

The Mach number is the ratio of the velocity of the shock (relative to the unshocked medium) to the sound speed in the unshocked medium. Using the Rankine-Hugoniot conditions, we may solve for the density, pressure, and temperature jumps in terms of M_1.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} = \frac{u_1}{u_2} \tag{3.43}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \tag{3.44}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma-1)M_1^2+2][2\gamma M_1^2-(\gamma-1)]}{(\gamma+1)^2 M_1^2} \ . \tag{3.45}$$

A strong shock is defined as one that is highly supersonic, with $M_1 \gg 1$. For a strong shock,

$$\frac{\rho_2}{\rho_1} \approx \frac{\gamma+1}{\gamma-1} \tag{3.46}$$

$$P_2 \approx \frac{2}{\gamma+1}\rho_1 u_1^2 \tag{3.47}$$

$$T_2 \approx \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2 \ . \tag{3.48}$$

Thus, no matter how strong the shock is, the ratio ρ_2/ρ_1 has a finite value; for a monatomic gas, with $\gamma = 5/3$, the ratio is $\rho_2/\rho_1 = 4$. However, a strong shock is very efficient at converting the bulk kinetic energy of the upstream gas ($\sim \rho_1 u_1^2$) into thermal energy.

A weak shock is defined as one that is barely supersonic, with $M_1 - 1 = \epsilon \ll 1$. For a weak shock,

$$\frac{\rho_2}{\rho_1} \approx 1 + \frac{4}{\gamma+1}\epsilon \tag{3.49}$$

$$\frac{P_2}{P_1} \approx 1 + \frac{4\gamma}{\gamma+1}\epsilon \tag{3.50}$$

$$\frac{T_2}{T_1} \approx 1 + \frac{4(\gamma-1)}{\gamma+1}\epsilon \ . \tag{3.51}$$

Generally speaking, a shock converts supersonic gas ($M_1 > 1$) into subsonic gas (in the shock's frame of reference). It *increases* density, *decreases* bulk velocity (relative to the shock front), *increases* pressure, and *increases* temperature.

The conversion of bulk kinetic energy to random thermal energy occurs by dissipation within the shock layer itself. Within the shock layer, a jump in velocity of $\Delta u \sim a_1 \sim (kT/m)^{1/2}$ occurs over a length $\Delta x \sim \lambda \sim 1/(n\sigma)$. The xx component of the viscous stress tensor within the shock is

$$\pi_{xx} = (\frac{4}{3}\pi + \beta)\frac{\Delta u}{\Delta x} \sim \frac{\sqrt{mkT}}{\sigma}\frac{\sqrt{kT}}{\sqrt{m}}(n\sigma) \sim nkT \ . \tag{3.52}$$

The increase in the specific entropy as the gas crosses the shock front is $s_2 - s_1 = c_P \ln(T_2/T_1) - (k/m)\ln(P_2/P_1)$. For an extremely strong shock ($M_1 \to \infty$), the entropy increase is $s_2 - s_1 \propto \ln M_1$.

So far, we've been dealing with **normal shocks**; that is, shocks in which the the velocity vector $\vec{u}_1$ of the upstream, unshocked gas is perpendicular to the shock front. However, some shocks are better described as **oblique shocks**, in which $\vec{u}_1$ approaches the shock front at an angle other than 90 degrees. For instance, the shock waves associated with spiral arms of galaxies can be approximated as oblique shocks. An oblique shock is illustrated in Figure 3.3. Again, let us put ourselves into a frame of reference that is moving along with a plane parallel shock front. This time, however, the unshocked gas is flowing into the shock at an angle ϕ with respect to the plane of the shock front. Thus, the bulk velocity u_1 can be broken down into a perpendicular component

$$u_{\perp 1} = u_1 \sin\phi \tag{3.53}$$

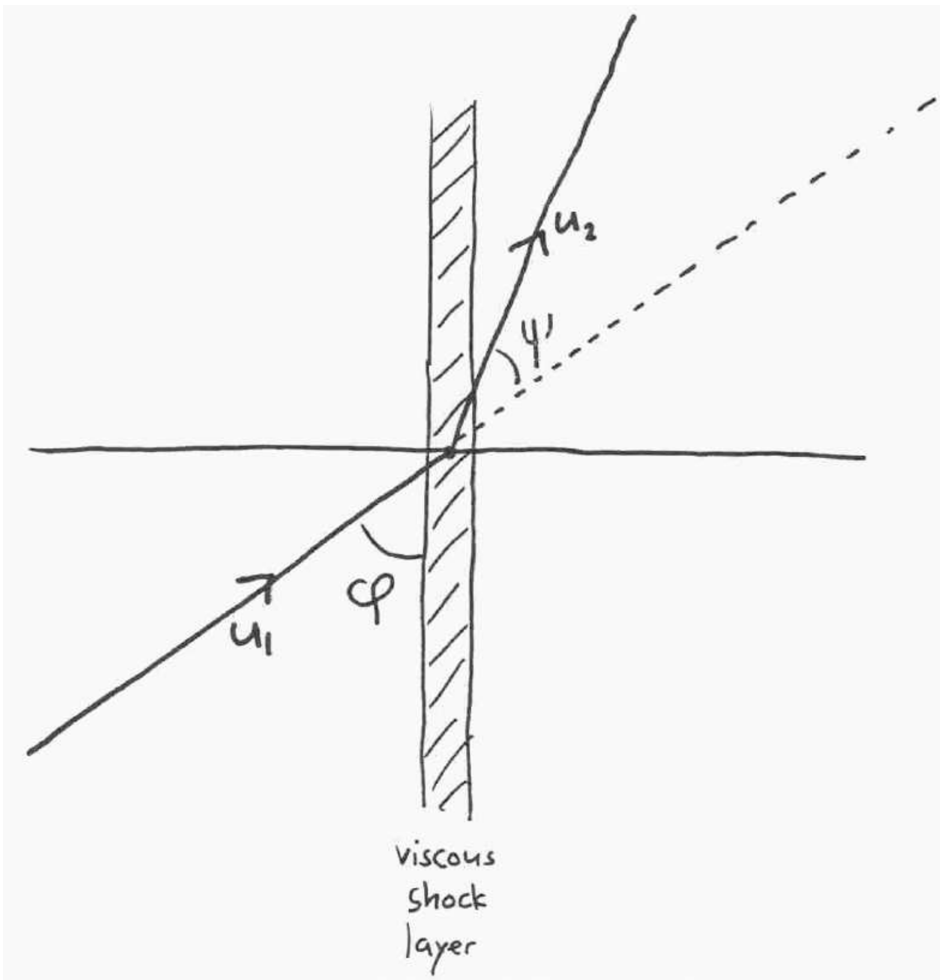

Figure 3.3: The geometry of an oblique shock; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

and a parallel component

$$u_{\parallel 1} = u_1 \cos\phi \ , \tag{3.54}$$

which are perpendicular and parallel, respectively, to the shock front. The postshock velocity $\vec{u}_2$ is rotated through an angle ψ away from $\vec{u}_1$ and towards the shock front. The perpendicular and parallel components of $\vec{u}_2$ are

$$u_{\perp 2} = u_2 \sin(\phi - \psi) \tag{3.55}$$

and

$$u_{\parallel 2} = u_2 \cos(\phi - \psi) \ . \tag{3.56}$$

The parallel component of the velocity is unchanged by passage through the shock. Thus,

$$u_1 \cos\phi = u_2 \cos(\phi - \psi) \ . \tag{3.57}$$

The Rankine-Hugoniot jump conditions for the perpendicular component are

$$\rho_1 u_{\perp 1} = \rho_2 u_{\perp 2} \tag{3.58}$$

$$\rho_1 u_{\perp 1}^2 = \rho_2 u_{\perp 2}^2 \tag{3.59}$$

$$\frac{1}{2} u_{\perp 1}^2 + \varepsilon_1 + \frac{P_1}{\rho_1} = \frac{1}{2} u_{\perp 2}^2 + \varepsilon_2 + \frac{P_2}{\rho_2} \ . \tag{3.60}$$

In an oblique shock, $u_\perp$ obeys the same relations as u does in a normal shock. From our previous encounter with the Rankine-Hugoniot jump conditions, we know that the ratio of the perpendicular velocities is

$$\frac{u_{\perp 2}}{u_{\perp 1}} = \frac{2 + (\gamma - 1) M_1^2 \sin^2\phi}{(\gamma + 1) M_1^2 \sin^2\phi} \ , \tag{3.61}$$

where $M_1 = u_1/a_1$ is the Mach number of the upstream, unshocked flow.

Since the parallel component of $\vec{u}$ is conserved and the perpendicular component is decreased, the velocity vector is refracted away from the normal to the shock front. Computing the actual value of the angle ψ through which it is refracted cumbersome in the general case. Combining the equations for the perpendicular and parallel components of the velocity, we find

$$\tan\psi = \frac{1 - \cos 2\phi - 2/M_1^2}{\gamma - \cos 2\phi + 2/M_1^2} \frac{\sin 2\phi}{1 - \cos 2\phi} \ . \tag{3.62}$$

In the case of a strong shock, the relation between ϕ and ψ takes the simpler form

$$\tan\psi = \frac{2\tan\phi}{(\gamma + 1) + (\gamma - 1)\tan^2\phi} \ . \tag{3.63}$$

The maximum value $\psi_{\rm m}$, in this case, is given by the relation

$$\tan\psi_{\rm m} = \left(\frac{1}{\gamma^2 - 1}\right)^{1/2} \quad , \tag{3.64}$$

which occurs when $\tan^2\phi = (\gamma+1)/(\gamma-1)$. For a monatomic gas ($\gamma = 5/3$) the maximum angle of refraction is $\psi_{\rm m} = 36.9°$, which occurs when $\phi = 63.4°$. Even an arbitrarily strong shock can't divert the flow through an angle of 90°. If a blunt piston, therefore, is shoved through a gas at supersonic speeds, then a *detached* bow shock must form ahead of the piston.