Chapter 4

Radiative & Magnetohydrodynamic Shocks

I have been dealing, so far, with non-radiative shocks. Since, as we have seen, a shock raises the density and temperature of the gas, it is quite likely, under astrophysical conditions, that the post-shock gas will be highly radiative.

Consider, again, plane parallel shocks in which the gas flow is perpendicular to the shock front, as shown in Figure 4.1. A gas with density ρ_1 and pressure P_1 flows into the shock front with a velocity u_1 relative to the front. The pre-shock gas is assumed to be in thermal equilibrium, and is not radiating energy. The gas passes through the thin shock layer; immediately downstream, it has density ρ_2, pressure P_2, and bulk velocity u_2 given by the Rankine-Hugoniot jump conditions. So far, everything is the same as in the nonradiative shock. Now, however, we assume that there is an optically thin **radiative relaxation layer** downstream of the shock front, in which the cooling function $L(\rho, T)$ is greater than zero.[1]

Within the radiative relaxation layer, the conservation equations are

$$\rho u = \text{constant} \tag{4.1}$$

$$\rho u^2 + P = \text{constant} \tag{4.2}$$

$$\rho u \frac{d\varepsilon}{dx} = -P\frac{du}{dx} - \rho L \ . \tag{4.3}$$

For an ideal gas, the energy equation reduces to

$$\frac{u}{\gamma - 1}\frac{dP}{dx} + \frac{\gamma}{\gamma - 1}P\frac{du}{dx} = -\rho L \tag{4.4}$$

[1]The cooling function L has units of energy per unit mass per unit time.

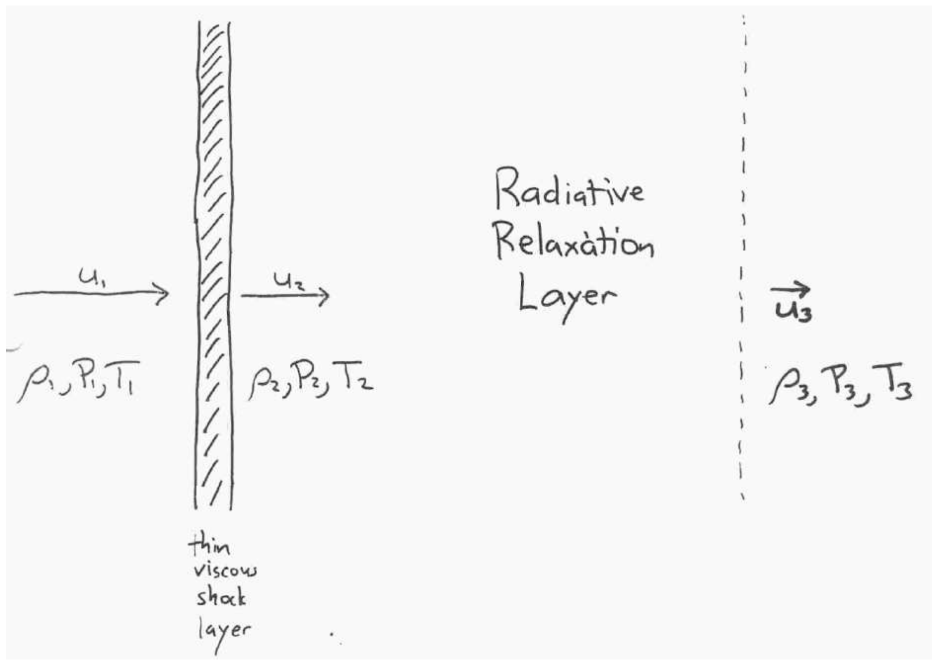

Figure 4.1: The geometry of a radiative plane parallel shock; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

or, using the continuity and momentum equations,

$$\frac{a^2 - u^2}{\gamma - 1}\frac{du}{dx} = -L(\rho, T) \ . \tag{4.5}$$

The cooling function for a very hot ionized gas, as an example, is dominated by bremsstrahlung, for which $L \propto \rho T^{1/2}$. At lower temperatures, the cooling may be dominated by radiative recombination, or line radiation, or some other process. Computing the exact values of ρ, P, T, and u in the radiative relaxation layer is usually done numerically, with tabulated values of L (or a functional fit). Without resorting to a computer, we can still derive the qualitative behavior of the gas in the radiative relaxation layer.

As initial conditions, take the values of ρ_2, P_2, T_2 and u_2 immediately downstream of the shock front. The four equations

$$\rho u = \rho_2 u_2 \tag{4.6}$$

$$\rho u^2 + P(x) = \rho_2 u_2^2 + P_2 \tag{4.7}$$

$$\frac{1}{\gamma - 1}(\gamma P/\rho - u^2)\frac{du}{dx} = -L(\rho, T) \tag{4.8}$$

$$T = \frac{m}{k}\frac{P}{\rho} \tag{4.9}$$

tell us the physical conditions within the radiative relaxation layer. Since the post-shock gas is subsonic ($u^2 < \gamma P/\rho$) and since $L > 0$, equation (4.8) indicates that the flow velocity u decreases as you move through the radiative relaxation layer away from the shock front. Since u decreases, equation (4.6) tells us that ρ increases. From equations (4.6) and (4.7), the pressure is

$$P = P_2 + \rho_2 u_2^2 (1 - u/u_2) \ . \tag{4.10}$$

For a strong shock $P_2 > \rho_2 u_2^2$, and the pressure increases only slightly as the gas flows through the radiative relaxation layer. The temperature, in terms of u, is

$$T = T_2 \left(\frac{u}{u_2}\right) \left[1 + \frac{\rho_2 u_2^2}{P_2}(1 - u/u_2)\right] \approx T_2 \left(\frac{u}{u_2}\right) \ . \tag{4.11}$$

For a strong shock, $T \propto u$ and the temperature decreases in the radiative relaxation layer. The gas will approach a steady state equilibrium in which $L = 0$, and the density, pressure, temperature, and bulk velocity have the constant values ρ_3, P_3, T_3, and u_3.

Frequently, the final temperature T_3 is the same as the initial value T_1. A radiative shock for which this equality holds true is called an **isothermal shock**. The jump conditions for an isothermal shock are

$$\rho_3 u_3 = \rho_1 u_1 \tag{4.12}$$
$$\rho_3 u_3^2 + P_3 = \rho_1 u_1^2 + P_1 \tag{4.13}$$
$$T_3 = T_1 \ . \tag{4.14}$$

Making use of the equation of state $P = \rho kT/m$, and using the "isothermal sound speed" $a_T \equiv (kT_1/m)^{1/2}$, we find that the increase in density is

$$\frac{\rho_3}{\rho_1} = \left(\frac{u_1}{a_T}\right)^2 = M_T^2 \ , \tag{4.15}$$

where M_T is the upstream isothermal Mach number. The bulk velocity is decreased by a factor $u_3/u_1 = 1/M_T^2$. The properties of a $M_T = 2$ isothermal shock are displayed in Figure 4.2. Radiative isothermal shocks can achieve arbitrarily high compression as the Mach number approaches infinity. Supernova remnants in the early stages of their expansion are surrounded by nonradiative shocks. However, as the spherical shocks expand and slow down, the postshock temperatures drop below $\sim 10^4\,\mathrm{K}$. At this point, line emission kicks in, and the cooling becomes much more efficient. Thus, highly evolved supernova remnants (such as the Cygnus Loop) have structures with very high density contrasts.

The effectiveness of cooling is indicated by the cooling time $t_T \sim \varepsilon/L$. At high temperatures, $T \sim 10^7\,\mathrm{K}$, the cooling time is $t_T \sim 6\,\mathrm{Myr}\,(1\,\mathrm{cm}^{-3}/n_H)$. At a temperature of $T \sim 8000\,\mathrm{K}$, cooling is much more efficient, and $t_T \sim 0.02\,\mathrm{Myr}\,(1\,\mathrm{cm}^{-3}/n_H)$. At lower temperatures, in the range $50\,\mathrm{K} \lesssim T \lesssim 600\,\mathrm{K}$, the cooling time is $t_T \sim 0.2\,\mathrm{Myr}\,(1\,\mathrm{cm}^{-3}/n_H)$.

It is now time to consider **magnetohydrodynamics** (or MHD). MHD is the study of the motions of a conducting fluid in the presence of magnetic fields. Magnetic field strengths encountered in astrophysics range from $10^{-6}\,\mathrm{G}$ in the hot ISM to $10^{12}\,\mathrm{G}$ on the surface of a neutron star. Magnetic fields have a significant effect on the dynamics of astrophysical gases.

A good place to start our discussion of MHD is by writing down Maxwell's equations for the electric field $\vec{E}$ and the magnetic flux $\vec{B}$.

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e \tag{4.16}$$

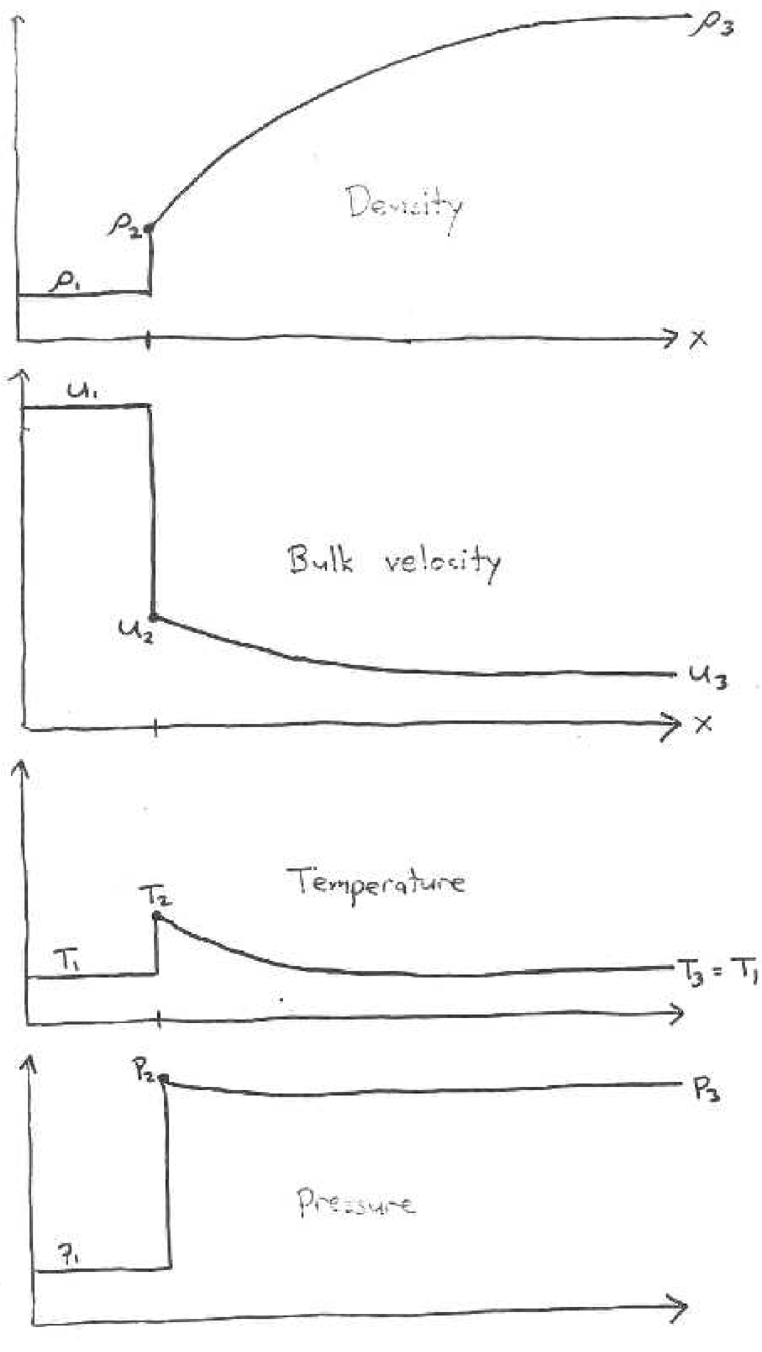

Figure 4.2: The density, bulk velocity, temperature, and pressure of material passing through a Mach 2 isothermal shock.

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{4.17}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4.18}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \ . \tag{4.19}$$

In the above equations, ρ_e is the charge density and $\vec{j}_e$ is the current density. In nonrelativistic systems, the term $c^{-1}\partial\vec{E}/\partial t$ in equation (4.19) may be ignored.

In an ionized gas, the current, electric field, and magnetic flux are related through the expression

$$\vec{E} = \frac{1}{\sigma}\vec{j}_e - \frac{1}{c}\vec{u} \times \vec{B} \ , \tag{4.20}$$

where σ is the electrical conductivity. Using equation (4.19) to eliminate j_e,

$$\vec{E} = \frac{c}{4\pi\sigma}\vec{\nabla} \times \vec{B} - \frac{1}{c}\vec{u} \times \vec{B} \ . \tag{4.21}$$

Substituting this result into equation (4.17), we find the basic equation for the evolution of $\vec{B}$:

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{u}) = -\vec{\nabla} \times (\eta \vec{\nabla} \times \vec{B}) \ . \tag{4.22}$$

The quantity η is the **electrical resistivity**,

$$\eta \equiv \frac{c^2}{4\pi\sigma} \ . \tag{4.23}$$

If η is constant throughout the gas, equation (4.23) may be rewritten in the form

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{u}) = \eta \nabla^2 \vec{B} \ , \tag{4.24}$$

where I have made use of the fact that $\vec{\nabla} \cdot \vec{B} = 0$. If $\vec{u} = 0$, then the above equation represents the *diffusion* of the vector field $\vec{B}$, with a diffusion constant η.

The electric and magnetic fields exert force on the charged particles that make up an ionized gas. For a completely ionized gas, with ions of charge Ze, the mean force per unit volume is

$$\vec{f}_L = Zen_i(\vec{E} + \frac{1}{c}\vec{u}_i \times \vec{B}) - en_e(\vec{E} + \frac{1}{c}\vec{u}_e \times \vec{B}) \ . \tag{4.25}$$

The ions have a number density n_i and a bulk velocity $\vec{u}_i$; the electrons have a number density n_e and a bulk velocity $\vec{u}_e$. Since maintaining charge neutrality requires that $Zen_i - en_e = 0$, and the current density is given by the relation $\vec{j}_e = Zen_i\vec{u}_i - en_e\vec{u}_e$, the Lorenz force per unit volume is

$$\begin{aligned} \vec{f}_L &= \frac{1}{c}\vec{j}_e \times \vec{B} && (4.26) \\ &= \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} \ . && (4.27) \end{aligned}$$

Using the appropriate identity from vector algebra,

$$\vec{f}_L = \frac{1}{4\pi}(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right) \ . \qquad (4.28)$$

The first term on the right hand side represents a 'magnetic tension'; it exists only if the magnetic field lines are curved, and acts in such a way as to straighten them out. The second term on the right hand side represents the gradient of the 'magnetic pressure', $P_m = B^2/(8\pi)$.

The magnetic field also adds a heating term to the internal energy conservation equation. An electric current j_e passing through a medium of resistivity η converts electromagnetic energy to heat energy at the rate

$$P = \frac{4\pi\eta}{c^2} j_e^2 = \frac{\eta}{4\pi}|\vec{\nabla} \times \vec{B}|^2 \ . \qquad (4.29)$$

For the sake of monumental completeness, let's write down the basic equations of MHD in all their glory.

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho\vec{\nabla} \cdot \vec{u} && (4.30) \\ \rho\frac{D\vec{u}}{Dt} &= -\vec{\nabla}P + \vec{\nabla} \cdot \overleftrightarrow{\pi} - \rho\vec{\nabla}\Phi + \vec{f}_{\rm rad} + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} && (4.31) \\ \rho\frac{D\varepsilon}{Dt} &= -P\vec{\nabla} \cdot \vec{u} - \vec{\nabla} \cdot \vec{F} + \psi + \Gamma - \Lambda + \frac{\eta}{4\pi}|\vec{\nabla} \times \vec{B}|^2 && (4.32) \\ \nabla^2\Phi &= 4\pi G\rho && (4.33) \\ \frac{\partial\vec{B}}{\partial t} &+ \vec{\nabla} \times (\vec{B} \times \vec{u}) = -\vec{\nabla} \times (\eta\vec{\nabla} \times \vec{B}) && (4.34) \\ \vec{\nabla} \cdot \vec{B} &= 0 && (4.35) \\ \varepsilon &= \frac{1}{\gamma - 1}\frac{P}{\rho} \ . && (4.36) \end{aligned}$$

Now that we've taken the trouble to write down these complicated equations, we will ruthlessly simplify them. First we will ignore viscosity and heat conduction, setting $\overleftrightarrow{\pi} = 0$, $\psi = 0$, and $\vec{F} = 0$. We will ignore gravity, and set $\vec{\nabla}\Phi = 0$. We will ignore radiative effects, and set $\vec{f}_{\rm rad} = 0$ and $\Gamma - \Lambda = 0$. We will ignore diffusion of the magnetic field, and set $\eta = 0$.

All of these simplifications will make it possible for us to consider a steady-state planar shock in the presence of a magnetic field. If a planar shock front exists, then the velocity vector $\vec{u}$ and the magnetic field vector $\vec{B}$ can both be broken down into components perpendicular to the shock front (designated by the subscript $\perp$) and parallel to the shock front (designated by the subscript $\|$). Quantities measured just upstream of the shock will be designated by the subscript '1', as usual, and quantities measured just downstream of the shock will be designated by the subscript '2'. The jump conditions across the shock are:

$$\begin{aligned}
\rho_1 u_{\perp 1} &= \rho_2 u_{\perp 2} && (4.37)\\
\rho_1 u_{\perp 1}^2 + P_1 + \frac{B_{\|1}^2}{8\pi} &= \rho_2 u_{\perp 2}^2 + P_2 + \frac{B_{\|2}^2}{8\pi} && (4.38)\\
\rho_1 u_{\perp 1} u_{\|1} - \frac{B_{\perp 1} B_{\|1}}{4\pi} &= \rho_2 u_{\perp 2} u_{\|2} - \frac{B_{\perp 2} B_{\|2}}{4\pi} && (4.39)\\
\rho_1 u_{\perp 1}\left(\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} + \frac{u_1^2}{2}\right) &- \frac{B_{\|1}}{4\pi}(B_{\perp 1} u_{\|1} - B_{\|1} u_{\perp 1}) = && (4.40)\\
& \rho_2 u_{\perp 2}\left(\frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} + \frac{u_2^2}{2}\right) - \frac{B_{\|2}}{4\pi}(B_{\perp 2} u_{\|2} - B_{\|2} u_{\perp 2}) &&\\
B_{\perp 1} u_{\|1} - B_{\|1} u_{\perp 1} &= B_{\perp 2} u_{\|2} - B_{\|2} u_{\perp 2} && (4.41)\\
B_{\perp 1} &= B_{\perp 2} && (4.42)
\end{aligned}$$

Note that the component of the velocity parallel to the shock is no longer conserved. The discontinuity in $u_\|$ occurs because there is a current within the shock front of strength $(c/4\pi)(B_{\|2} - B_{\|1})$, which increases $u_\|$ by a factor

$$u_{\|2} - u_{\|1} = \frac{B_\perp}{4\pi\rho u_\perp}(B_{\|2} - B_{\|1}) \ . \qquad (4.43)$$

The most mathematically tractable MHD shock is a normal shock ($u_\| = 0$, $u_\perp = u$) in which the magnetic field is parallel to the shock front ($B_\perp = 0$, $B_\| = B$). In this case, the jump relations simplify to the form

$$\rho_1 u_1 = \rho_2 u_2 \qquad (4.44)$$

$$\rho_1 u_1^2 + P_1 + \frac{B_1^2}{8\pi} = \rho_2 u_2^2 + P_2 + \frac{B_2^2}{8\pi} \tag{4.45}$$

$$\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} + \frac{1}{2}u_1^2 + \frac{B_1}{4\pi\rho_1} = \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} + \frac{1}{2}u_2^2 + \frac{B_2^2}{4\pi\rho_2} \tag{4.46}$$

$$B_1 u_1 = B_2 u_2 \tag{4.47}$$

Using these four equations, we can solve for the density jump ρ_2/ρ_1 in terms of the upstream Mach number M_1 and the ratio of the magnetic pressure to the thermal pressure,

$$\alpha_1 \equiv \frac{B_1^2}{8\pi P_1} \ . \tag{4.48}$$

After discarding the trivial solution $\rho_2 = \rho_1$, the four jump relations simplify to the quadratic relation

$$2(2-\gamma)\alpha_1 \left(\frac{\rho_2}{\rho_1}\right)^2 + \gamma[(\gamma-1)M_1^2 + 2(\alpha_1+1)] \left(\frac{\rho_2}{\rho_1}\right) - \gamma(\gamma+1)M_1^2 = 0 \ . \tag{4.49}$$

When the magnetic pressure is relatively insignificant ($\alpha_1 \ll M_1^2$) the change in density is approximately

$$\frac{\rho_2}{\rho_1} = y_0 \left[1 - \frac{4}{\gamma}\frac{\gamma + M_1^2}{[2+(\gamma-1)M_1^2]^2}\alpha_1\right] \ , \tag{4.50}$$

where y_0 is the density ratio in the absence of a magnetic field. The presence of a magnetic field thus tends to decrease the density jump from what it would be in the absence of magnetism. In fact, if we again examine the quadratic equation for ρ_2/ρ_1, we see that $\rho_2 > \rho_1$ when

$$M_1^2 > M_{\rm cr}^2 \equiv 1 + \frac{2}{\gamma}\alpha_1 \ . \tag{4.51}$$

The existence of a magnetic field permits you to have supersonic motions ($M > 1$) without the formation of shocks. Magnetosonic waves (fluctuations in the combined magnetic and gas pressure) can travel ahead of the shock front. Fluctuations in the gas pressure have a characteristic speed $a \propto (P/\rho)^{1/2}$. Similarly, fluctuations in the magnetic pressure have a characteristic speed, called the **Alfven velocity**, $u_A \equiv B/(4\pi\rho)^{1/2} \propto (P_m/\rho)^{1/2}$. Magnetosonic waves that travel in a direction perpendicular to $\vec{B}$ have a phase velocity $u = (a^2 + u_A^2)^{1/2} = a[1 + \alpha(2/\gamma)]^{1/2}$.