Chapter 7

Basic Turbulence

The universe is a highly turbulent place, and we must understand turbulence if we want to understand a lot of what's going on. Interstellar turbulence causes the "twinkling" of radio sources, just as turbulence in the earth's atmosphere causes the twinkling of stars. Turbulence in stellar atmospheres strongly modifies the structure of the star; in fact, the turbulent heat flux can be larger than the conductive heat flux $\vec{F}$ and the turbulent stress tensor can be larger than the viscous stress tensor $\overleftrightarrow{\pi}$. Turbulent motions are seen in HII regions, in molecular clouds, and in jets.

The first difficulty in dealing with turbulence is providing an adequate definition. According to Webster's New Collegiate Dictionary, turbulence is "departure in a fluid from a smooth flow". Turbulence is characterized by the presence of irregular eddying motions – that is, motions in which the **vorticity** $\vec{\omega} \equiv \vec{\nabla} \times \vec{u}$ is non-zero. An image of turbulent flow in a terrestrial laboratory is given in Figure 7.1. Usually, a turbulent flow has a spectrum of eddy sizes.[1]

In a turbulent flow, the fluid velocity $\vec{u}(\vec{x}, t)$ at a fixed point $\vec{x}$ varies with time in a nearly random manner, as shown in the data of Figure 7.2. From an alternative viewpoint, the velocity at a fixed time t varies with position in a nearly random manner. Turbulence is a **chaotic** process; a small change in the initial conditions $\vec{u}(\vec{x}, t_0)$ results in a large change in the conditions at a later time t. Since we cannot examine the development of the turbulent velocity field in a deterministic manner, we are reduced to studying the statistical properties of the turbulence.

[1] As it says in the poem by Lewis F. Richardson, 'Big whirls have little whirls / Which feed on their velocity, / And little whirls have lesser whirls / And so on to viscosity.'

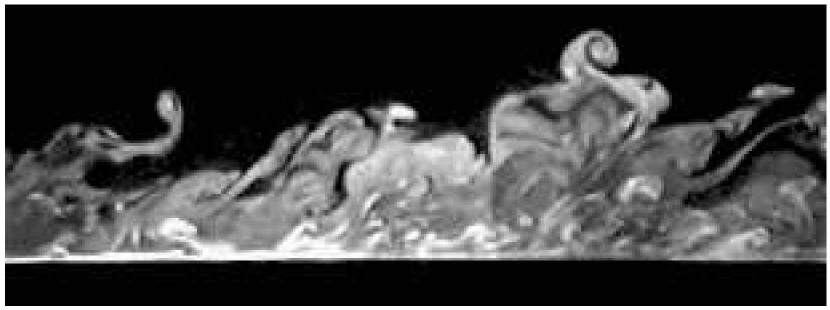

Figure 7.1: Laser-induced fluorescence image of an incompressible turbulence boundary layer. [University of Iowa Fluids Laboratory]

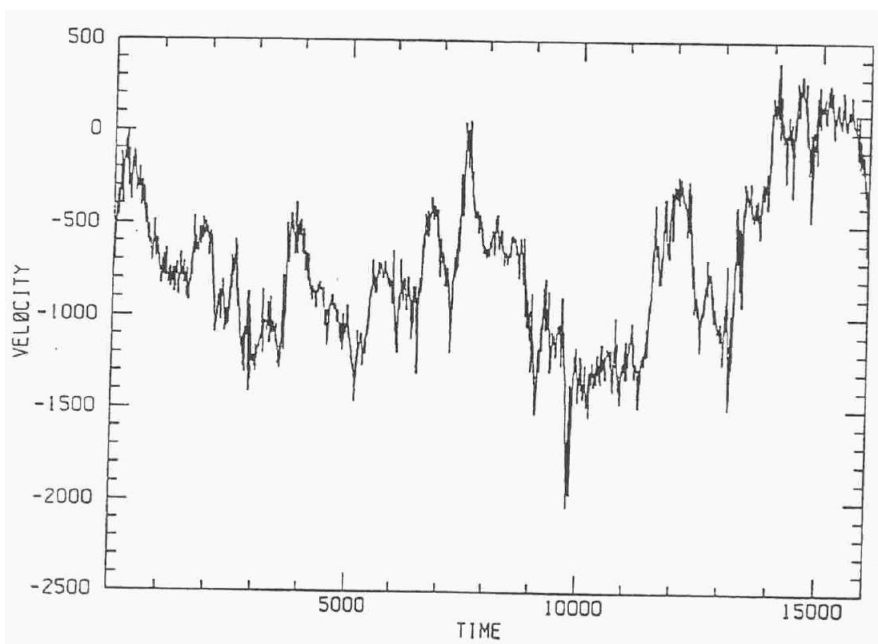

Figure 7.2: Turbulent velocities within the Earth's atmosphere, measured 18 m above the ground. Time and velocity are in arbitrary units.

A flow that is not turbulent, in which the velocity varies smoothly and predictably in space and time, is known as a **laminar** flow. Many flows can be broken down into a laminar and a turbulent component. Note that the random *turbulent* component of a fluid's motion is distinct from the random *thermal* component. The thermal component consists of the random Brownian motion of the individual gas particles; each particle moves in a straight line for a distance $\sim \lambda$ before an encounter with another particle sends it off in a different direction. The turbulent component consists of the eddying motion of macroscopic fluid elements; the eddies have a range of sizes, but all eddies are very much larger than λ.

By distinguishing between the laminar component and the turbulent component, we are separating the bulk velocity $\vec{u}$ of a gas into two parts:

$$\vec{u}(\vec{x}) = \vec{U}(\vec{x}) + \vec{u}'(\vec{x}) \ . \tag{7.1}$$

where the laminar component is $\vec{U} = \langle \vec{u} \rangle$ and the turbulent component is $\vec{u}' = \vec{u} - \langle \vec{u} \rangle$. When I use the symbol $\langle \vec{u} \rangle$, I am referring to the average value of $\vec{u}$ – but what does it mean, in this context, to take the average of the velocity? In practice, what we must do is calculate the spatial average within a volume V,

$$\langle \vec{u} \rangle = \frac{1}{V} \int_V \vec{u} d^3x \ , \tag{7.2}$$

or the time average during an interval τ,

$$\langle \vec{u} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \vec{u} dt \ . \tag{7.3}$$

The **Taylor hypothesis** states that for fully developed turbulence, the spatial average and the time average are equivalent.

Now, after removing the laminar velocity $\vec{U}$, we are left with the turbulent velocity $\vec{u}'$. We will make things easier for ourselves by placing some restrictions on the form of the turbulent velocity field. First, we will assume that the turbulence is **homogeneous** and **isotropic**; that is, the statistical properties of $\vec{u}'$ are independent of position and direction. In addition, we assume that the turbulence is **incompressible**; that is, the density of the turbulent fluid is the same everywhere. This assumption is an extraordinarily poor one for the interstellar medium, in which the density varies wildly from place to place. However, the case in which the fluid is incompressible has the great advantage of simplicity; if we assume incompressibility, we can actually find a few analytic results! These results will give us some insight into the physics of the problem, which we can then apply to the more realistic case of a self-gravitating, radiating, compressible fluid.

In an incompressible fluid, the continuity equation for the turbulent flow simplifies to the form

$$\vec{\nabla} \cdot \vec{u}' = 0. \tag{7.4}$$

The equation for momentum conservation is

$$\frac{\partial \vec{u}'}{\partial t} + (\vec{u}' \cdot \vec{\nabla})\vec{u}' = \frac{1}{\rho}\vec{\nabla} P + \nu \nabla^2 \vec{u}' \ . \tag{7.5}$$

I am assuming that the fluid exhibits Newtonian viscosity, and that the kinematic viscosity, $\nu \equiv \mu/\rho$, is constant. The kinematic viscosity of air is $\nu \approx 0.15\,\mathrm{cm}^2\,\mathrm{s}^{-1}$ at sea level and at room temperature. For neutral atomic hydrogen, the kinematic viscosity is

$$\nu \sim 4 \times 10^{21}\,\mathrm{cm}^2\,\mathrm{s}^{-1} \left(\frac{T}{10^4\,\mathrm{K}}\right)^{1/2} \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1} \ . \tag{7.6}$$

The kinematic velocity is approximately equal to the sound speed in a fluid times the mean free path λ.

We can define a **velocity correlation tensor** for the turbulent velocity in the following manner:

$$R_{ij}(r) \equiv \langle u'_i(\vec{x}) u'_j(\vec{x} + \vec{r}) \rangle \ . \tag{7.7}$$

Since the turbulent velocity field is homogeneous and isotropic (by assumption), the correlation tensor is a function only of the distance r between the two points and not on their location $\vec{x}$ within the velocity field. In an ideal universe, we could measure the velocity vector everywhere within the turbulent flow, subtract away the laminar component, and then compute the correlation tensor R_{ij} exactly. Unfortunately, this is difficult to do even inside a laboratory. Astronomers are even more badly hampered; usually the best they can get is a line-of-sight velocity profile, which adds together the radial velocities of all the matter along a given line of sight.

The trace of the velocity correlation tensor is

$$R(r) \equiv \sum_i R_{ii}(r) \ . \tag{7.8}$$

In the limit that $r \to 0$, $R(r) \to \langle |\vec{u}'|^2 \rangle$. The trace can be used to define a **correlation length**

$$\Lambda_t \equiv \frac{1}{R(0)} \int_0^\infty R(r)dr \ . \tag{7.9}$$

The correlation length, roughly speaking, is the size of the biggest eddies in the turbulent flow.

Since turbulence exists with a range of eddy sizes, it is frequently convenient to take the Fourier transform of the velocity field in order to consider the Fourier components of different wavenumber. The components of the velocity in Fourier space will be

$$\vec{u}_{\vec{k}} = \frac{1}{(2\pi)^3} \int \vec{u}'(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d^3r \ . \tag{7.10}$$

The Fourier transform of the velocity correlation tensor is the **energy spectrum tensor**,

$$\Phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int R_{ij}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d^3r \ . \tag{7.11}$$

The spectrum $\Phi_{ij}(\vec{k})$ tells how much kinetic energy is contained in eddies with wavenumber k. The tensors R_{ij} and Φ_{ij} both contain the same information about the field; which tensor you use depends merely on whether it is more convenient to work in real space or Fourier space.

In a homogeneous isotropic turbulent flow, it is possible, and also useful, to define an **energy spectrum function** $E(k)$ such that

$$E(k) \equiv 2\pi k^2 \sum_i \Phi_{ii}(\vec{k}) \ . \tag{7.12}$$

The total turbulent kinetic energy per unit mass is then

$$\frac{1}{2} \langle |\vec{u}'|^2 \rangle = \int_0^\infty E(k)dk \ . \tag{7.13}$$

In a Newtonian fluid, the rate at which the turbulent kinetic energy is dissipated by viscosity is

$$\epsilon_d = 2\nu \int_0^\infty k^2 E(k)dk \ . \tag{7.14}$$

Thus, the dissipation occurs at high wavenumbers – it is the smallest eddies present that dissipate the turbulent energy through viscosity.

For a turbulent flow to remain in a steady state, turbulent energy must be added at the largest scales at the same rate ϵ_d at which it is being dissipated at the smallest scales. If additional energy is not added, the turbulence will gradually decay. In the earth's atmosphere, for instance, the turbulence is maintained by the addition of solar energy, which adds energy to the atmosphere at the time-averaged rate $\epsilon_d \sim 10\,\mathrm{erg\,g^{-1}\,s^{-1}}$.

In 1941, Kolmogorov predicted that the velocity field in incompressible turbulent flows would be self-similar over a range of velocity scales.[2] The self-similarity, he stated, results from a dissipationless cascade of energy from large scales (the big whirls) to small scales (the little whirls). The average velocity difference between two points, in the Kolmogorov theory, is a function only of the distance r between the points and the energy dissipation rate per unit mass ϵ_d.

The fully developed turbulent medium is characterized by only two quantities, the mean rate of energy dissipation, ϵ_d, and the kinematic viscosity ν. The dimensionality of ϵ_d is energy/time/mass, or L^2T^{-3}, and the dimensionality of ν is L^2T^{-1}. We can combine these two quantities to find the length scale

$$l_K = \left(\frac{\nu^3}{\epsilon_d}\right)^{1/4} , \tag{7.15}$$

known as the Kolmogorov, or dissipational, length scale. The Kolmogorov length scale is the size of the smallest eddies in the fluid. Eddies smaller than the Kolmogorov scale rapidly dissipate their kinetic energy by viscous heating and disappear. The eddies of size l_K rotate with a velocity

$$u_K = (\nu\epsilon_d)^{1/4} , \tag{7.16}$$

and dissipate their energy in a time roughly equal to

$$\tau_K = \left(\frac{\nu}{\epsilon_d}\right)^{1/2} . \tag{7.17}$$

In the Earth's atmosphere, $\epsilon_d \sim 10\,\mathrm{cm^2\,s^{-3}}$ and $\nu \sim 0.1\,\mathrm{cm^2\,s^{-1}}$. Thus, the smallest eddies in the Earth's atmosphere have $l_K \sim 0.1\,\mathrm{cm}$, which is significantly longer than the mean free path $\lambda \sim 10^{-4}\,\mathrm{cm}$. The smallest eddies swirl around with a velocity $u_K \sim 1\,\mathrm{cm\,s^{-1}}$, dying away with a decay time of $\tau_K \sim 0.1\,\mathrm{s}$.

[2]His results were published in the Proceedings of the Soviet Academy of Sciences, and were initially neglected; Russians had other things to worry them during the year 1941.

Kolmogorov, in making his assumption of self-similar turbulence, saw that the power spectrum of the turbulence should have the form

$$E(k,t) = u_K^2 l_K E_*(l_K k) \ , \tag{7.18}$$

where E_* is a dimensionless function of the dimensionless wavenumber $l_K k$. The factor $u_K^2 l_K = \nu^{5/4}\epsilon_d^{1/4}$ in front gives the energy spectrum its proper dimensionality. When the wavenumber k of a turbulent element lies in the range $\Lambda_t^{-1} \ll k \ll l_K^{-1}$, then it is in the *inertial subrange*, in which negligible dissipation occurs, and the dominant energy process is the transfer of kinetic energy from large eddies to smaller eddies by inertial forces. In the inertial subrange, the power spectrum must be scale free, so that

$$E_*(l_K k) = \alpha (l_K k)^n \tag{7.19}$$

is a pure power law (with α being our old friend, the "factor of order unity").

Since viscous forces are negligible on this scale, the power spectrum $E(k)$ must also be independent of the value of the kinematic viscosity ν. Since $u_K^2 = \nu^{1/2}\epsilon_d^{1/2}$ and $l_K = \nu^{3/4}\epsilon_d^{-1/4}$, we find

$$E(k) = \nu^{1/2}\epsilon_d^{1/2}\nu^{3/4}\epsilon_d^{-1/4}\alpha\nu^{3n/4}\epsilon_d^{-n/4}k^n \ . \tag{7.20}$$

For the dependence of E on ν to vanish, we require $n = -5/3$, and

$$E(k) = \alpha\epsilon_d^{2/3}k^{-5/3} \ . \tag{7.21}$$

This spectrum, of the form $E \propto k^{-5/3}$, is referred to as the **Kolmogorov spectrum**, and is a pretty good fit for many turbulent flows on scales between the correlation length Λ_t and the Kolmogorov length l_K. A schematic plot of the Kolmogorov spectrum is shown in Figure 7.3.

If $u(l)$ is the velocity of eddies with size l, dimensional analysis tells us that

$$E(k) \sim l u(l)^2 \sim \epsilon_d^{2/3} l^{5/3} \tag{7.22}$$

and hence that

$$u(l) \sim \epsilon_d^{1/3} l^{1/3} = u_K (l/l_K)^{1/3} \ . \tag{7.23}$$

On scales larger than l_K the Reynolds number scales as $\mathrm{Re} \propto l^{4/3}$, so on large scales dissipation is negligible.

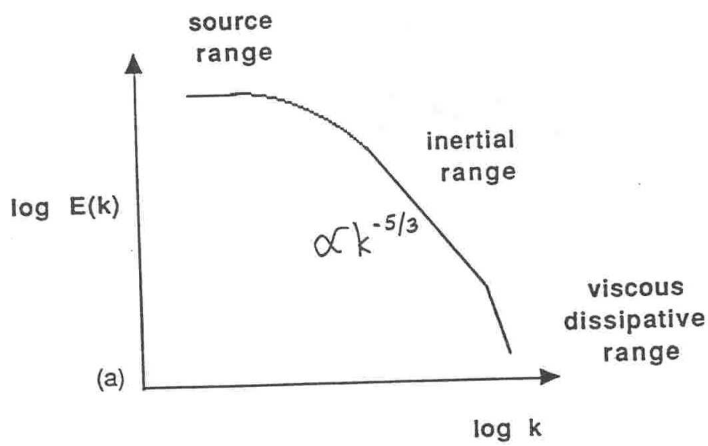

Figure 7.3: The Kolmogorov spectrum for incompressible turbulence.

In the Earth's atmosphere, the inertial range stretches from $l_K \sim 0.1\,\mathrm{cm}$ to the correlation length $\Lambda_t \sim 1\,\mathrm{km}$. Within that range, the typical turbulent velocity is

$$u(l) \approx 10\,\mathrm{cm\,s^{-1}} \left(\frac{l}{1\,\mathrm{m}}\right)^{1/3} . \tag{7.24}$$

The smallest eddies, with $l \sim l_K \sim 0.1\,\mathrm{cm}$, take 0.1 seconds to whirl around. The largest eddies, with $l \sim \Lambda_t \sim 1\,\mathrm{km}$, take 1000 seconds to whirl around. It is these turbulent eddies that cause 'twinkling' and 'seeing'.

Does the Kolmogorov spectrum apply (even approximately) to the interstellar medium, which is most distinctly *not* an incompressible fluid? For instance, it has been proposed that turbulence occurs within molecular clouds. The temperature of molecular clouds is usually about 10 K. If the emission lines in the spectra of molecular clouds are broadened only by the thermal motions, they should have a width of $\Delta v \sim 0.3\,\mathrm{km\,s^{-1}}$. In fact, the measured widths are greater than this value. For instance, in 1981, Richard Larson found that the relation

$$\Delta v \approx 1.1\,\mathrm{km\,s^{-1}} \left(\frac{l}{1\,\mathrm{pc}}\right)^{0.38} \tag{7.25}$$

gave a good fit to the measured velocity dispersions of molecular clouds. Larson's results are plotted in Figure 7.4. This observationally determined relation is close to the $\Delta v \propto l^{1/3}$ relation expected for a Kolmogorov spectrum of turbulence.

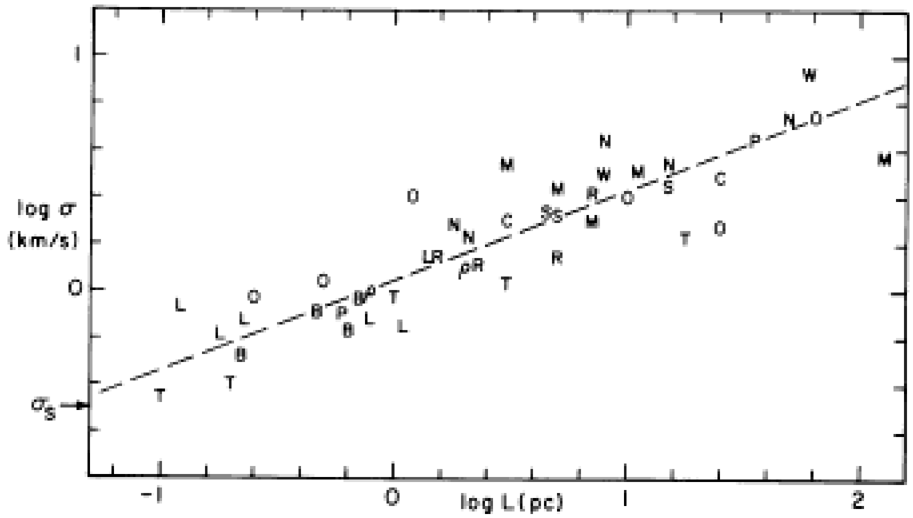

Figure 7.4: The internal velocity dispersion σ plotted versus the maximum linear size L of molecular clouds. (The different letter symbols represent different molecular cloud complexes.) [Larson, 1981, MNRAS, 194, 809]

If the relation were actually $\Delta v \sim 1\,\mathrm{km\,s^{-1}}(l/1\,\mathrm{pc})^{1/3}$, and the molecular gas were incompressible, that would imply $\epsilon_d = (\Delta v)^3/l \sim 3\times10^{-4}\,\mathrm{cm^2\,s^{-3}} \sim 2\times10^{-4}\,\mathrm{L_\odot/\,M_\odot}$. The kinematic viscosity of a dense molecular cloud is $\nu \sim 10^{17}\,\mathrm{cm^2\,s^{-1}}$, implying a Kolmogorov length $l_K \sim 4\times10^{13}\,\mathrm{cm} \sim 3\,\mathrm{AU}$. The velocity of the smallest eddies would be $u_K \sim 2\times10^3\,\mathrm{cm\,s^{-1}}$ and they would whirl around on a timescale $\tau_K \sim 2\times10^{10}\,\mathrm{s} \sim 600\,\mathrm{yr}$.

However, treating molecular clouds as if they were incompressible fluids with a Kolmogorov spectrum of turbulence is incredibly naïve. Most studies of molecular clouds give the relation

$$\Delta v \approx 1\,\mathrm{km\,s^{-1}}\left(\frac{l}{1\,\mathrm{pc}}\right)^{0.5} \tag{7.26}$$

for the line width on a scale l; the results from one study are shown in Figure 7.5. In addition, the average density of a molecular cloud scales as

$$n \approx 4000\,\mathrm{cm^{-3}}\left(\frac{l}{1\,\mathrm{pc}}\right)^{-1.2} . \tag{7.27}$$

Thus, the assumption of a constant density, incompressible fluid is a poor one. In addition, the turbulent velocities are larger than the sound speed

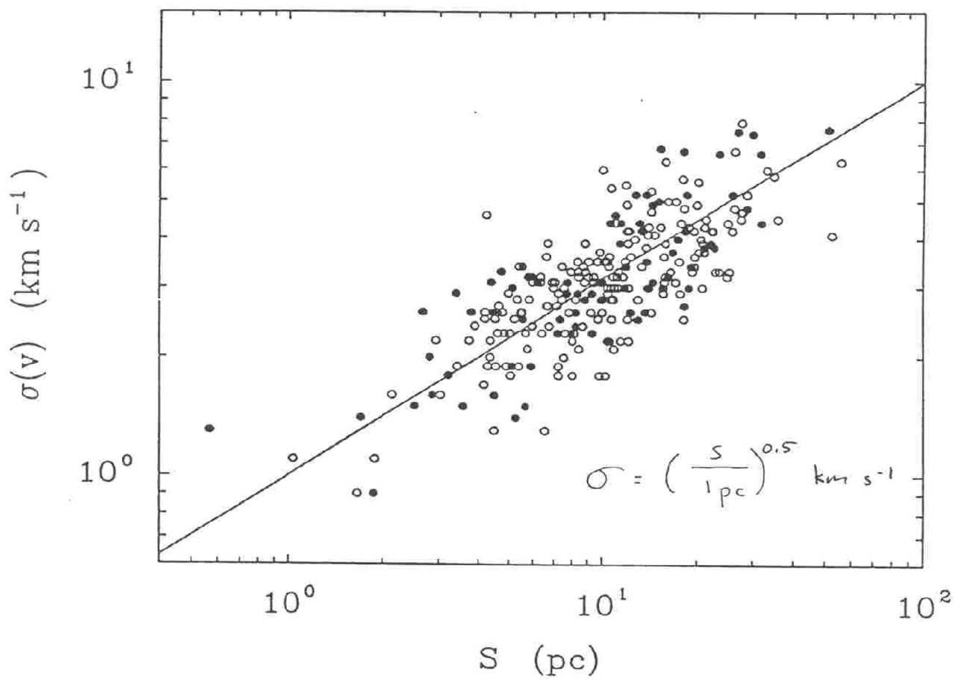

Figure 7.5: Velocity dispersion as a function of linear size for a sample of 273 molecular clouds within our galaxy. [Solomon et al., 1987, ApJ, 319, 730]

$a \sim 0.3\,\mathrm{km\,s^{-1}}$; therefore, it is highly likely that shocks will form. Furthermore, energy will be added to the molecular clouds not only on the correlation length Λ_t, but also on smaller scales (as the result of supernovae, stellar winds, and other processes). Also, in a realistic view of interstellar turbulence, the effects of gravity and magnetism must be taken into account. In view of these complications, people who study turbulence in the ISM must resort to numerical simulations. The simulation shown in Figure 7.6, for instance, shows the large density contrasts typical of realistic simulations of compressible turbulence.

Figure 7.6: Gas density from a 1024^3 adaptive mesh simulation of supersonic compressible turbulence (darker regions are denser). [Kritsuk et al., 2006, ApJ, 638, L25]