Chapter 9

Accretion Disks for Beginners

When the gas being accreted has high angular momentum, it generally forms an accretion disk. If the gas conserves angular momentum but is free to radiate energy, it will lose energy until it is on a circular orbit of radius $R_c = j^2/(GM)$, where j is the specific angular momentum of the gas, and M is the mass of the accreting compact object. The gas will only be able to move inward from this radius if it disposes of part of its angular momentum. In an accretion disk, angular momentum is transferred by viscous torques from the inner regions of the disk to the outer regions.

The importance of accretion disks was first realized in the study of binary stellar systems. Suppose that a compact object of mass M_c and a 'normal' star of mass M_s are separated by a distance a. The normal star (a main-sequence star, a giant, or a supergiant) is the source of the accreted matter, and the compact object is the body on which the matter accretes. If the two bodies are on circular orbits about their center of mass, their angular velocity will be

$$\vec{\Omega} = \left[\frac{G(M_c + M_s)}{a^3} \right]^{1/2} \hat{e} \ , \tag{9.1}$$

where $\hat{e}$ is the unit vector normal to the orbital plane.

In a frame of reference that is corotating with the two stars, the equation of momentum conservation has the form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} P - 2\vec{\Omega} \times \vec{u} - \vec{\nabla}\Phi_R \ , \tag{9.2}$$

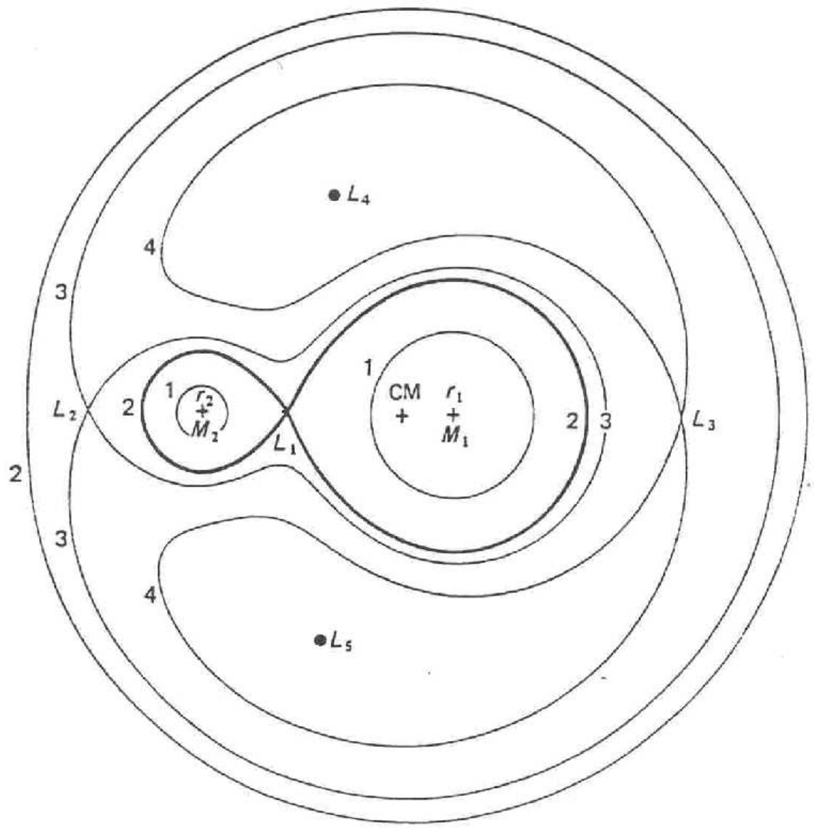

Figure 9.1: Sections in the orbital plane of equipotential surfaces, for a binary with $q = 0.2$. The five Lagrange points, L_1 through L_5, are shown.

where Φ_R is the Roche potential,

$$\Phi_R(\vec{r}) = -\frac{GM_c}{|\vec{r} - \vec{r}_1|} - \frac{GM_s}{|\vec{r} - \vec{r}_2|} - \frac{1}{2}|\vec{\Omega} \times \vec{r}|^2 \ . \tag{9.3}$$

The vectors $\vec{r}_1$ and $\vec{r}_2$ are the positions of the two stars in the rotating frame of reference; all distances are measured relative to the center of mass of the system.

The shapes of the equipotential surfaces (where $\Phi_R =$ constant) are dictated by the mass ratio $q \equiv M_s/M_c$. Figure 9.1 shows the equipotential surfaces for a binary system with $M_s/M_c = 0.2$. At very large distances, $r \gg a$, the equipotential surfaces are nearly spherical, and are centered on the center of mass of the binary system. At very small distances from the two stars, the equipotential surfaces form a pair of nearly spherical surfaces, centered on the two stars. It is at intermediate distances that the equipotential surfaces take on interesting shapes. Of particular interest in the context of accretion is the 'critical surface', which has a dumbbell shape, and consists of two **Roche lobes**, one for each star, which are connected at the **inner Lagrangian point** L_1. The L_1 point is a saddle-point in the potential; the gravitational acceleration at the L_1 point is equal to zero. If the normal star, in the process of stellar evolution, expands to fill its Roche lobe, then the gas at the L_1 point will be pushed, by its internal gas pressure, from the Roche lobe of the normal star (M_s) to the Roche lobe of the compact star (M_c). The mass transfer between the two components will change the period of the binary system. If the total mass and the orbital angular momentum of the system are constant, then the period of the system is $P \propto M_c^{-3} M_s^{-3}$. The period is minimized, given the constraint that $M_c + M_s$ is constant, when $M_c = M_s$. Kepler's law tells us that $a \propto M_c^{-2} M_s^{-2}$, which is also minimized when $M_c = M_s$. When the normal 'donor' star is more massive than the compact accretor ($M_s > M_c$), the loss of mass will decrease the separation a and will decrease the size of the Roche lobe of the normal star; thus, the mass loss will be naturally self-perpetuating. When $M_s < M_c$, however, the separation a will increase as the normal star loses mass, and thus the mass loss will tend to be cut off (unless evolutionary swelling continues to increase the size of the mass-losing star, or loss of angular momentum from the binary system shrinks the separation between the stars.)

The distance of the L_1 point from the center of the compact star is well approximated by the formula

$$\frac{b_1}{a} = 0.500 + 0.227 \log q \ , \tag{9.4}$$

so that even when the mass shed by the normal star crosses over the L_1 saddle point, it has a considerable distance to travel until it can collide with the compact star. Moreover, the gas that is passing through the L_1 point has considerable angular momentum. In a nonrotating frame of reference fixed to the compact star, the L_1 point is orbiting with a velocity $v \sim b_1 \Omega$. The accreting matter that is being squirted through the L_1 nozzle therefore has a bulk velocity $u_\perp \sim v \sim 2\pi b_1/P$ perpendicular to the line between the stars. Since the accreting matter is being squirted by pressure forces, the velocity of the gas parallel to the line between the stars will be $u_\parallel \sim a_s$, where a_s is the sound speed in the outermost envelope of the mass-losing star. The normal temperatures of stellar envelopes are $T \lesssim 10^5\,\mathrm{K}$, so typically you will have $u_\parallel \lesssim 30\,\mathrm{km\,s^{-1}}$. If the masses of the two stars are roughly equal, however, the use of Kepler's law tells us

$$u_\perp \sim 100\,\mathrm{km\,s^{-1}} \left(\frac{M_c}{\mathrm{M_\odot}}\right)^{1/3} \left(\frac{P}{1\,\mathrm{day}}\right)^{-1/3} . \tag{9.5}$$

In general, then, $u_\perp$ will be greater than $u_\parallel$. The specific angular momentum of the accreting gas will be $j = b_1^2(2\pi/P) = b_1^2(GM_c)^{1/2}(1+q)^{1/2}a^{-3/2}$. If it conserves angular momentum during infall, it will end up on a circular orbit of radius $R_c = (1+q)(b_1/a)^4 a$. For $q \gtrsim 0.5$, the circular radius will be roughly $R_c \sim 1\,\mathrm{R_\odot}(P/1\,\mathrm{day})^{2/3}$. Even for orbital periods as short as hours, this radius is larger than the radius of a white dwarf ($R \sim 0.01\,\mathrm{R_\odot}$), to say nothing of a neutron star ($R \sim 2 \times 10^{-5}\,\mathrm{R_\odot}$), or a solar mass black hole ($R \sim 4 \times 10^{-6}\,\mathrm{R_\odot}$).

Let us start our study of the dynamics of disks with the **thin disk approximation**; we will assume that the height H of the disk in the z direction is much smaller than the extent of the disk in the R direction. We will also assume that the disk is axisymmetric. If the mass of the disk is negligible compared to the mass of the central compact object, the angular velocity will have the Keplerian form

$$\Omega(R) = (GM/R^3)^{1/2} , \tag{9.6}$$

so that the circular velocity will be $u_\phi(R) = R\Omega(R) \propto R^{-1/2}$. In addition to the circular velocity u_ϕ, the matter of the disk will have a small radial drift velocity $u_R(R,t)$ that carries the gas inward or outward.

Consider the two thin annuli of matter on either side of the surface $R =$ constant. If the kinematic viscosity of the gas is ν, the viscous torque exerted on the inner annulus by the outer annulus is

$$T(R) = 2\pi\nu\Sigma R^3 \frac{d\Omega}{dR} , \tag{9.7}$$

where $\Sigma(R,t)$ is the **surface density** of the disk. For a disk in which Ω decreases with radius, $T(R)$ is negative.

The mass conservation equation for a thin disk takes the form

$$R\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma u_R) = 0 . \tag{9.8}$$

The conservation equation for angular momentum takes the form

$$R\frac{\partial}{\partial t}(\Sigma R^2\Omega) + \frac{\partial}{\partial R}(\Sigma u_R R^3\Omega) = \frac{1}{2\pi}\frac{\partial T}{\partial R} \ . \tag{9.9}$$

These two equations may be combined into a single equation

$$R\frac{\partial \Sigma}{\partial t} = -\frac{1}{2\pi}\frac{\partial}{\partial R}\left[\frac{1}{(R^2\Omega)'}\frac{\partial T}{\partial R}\right] \ . \tag{9.10}$$

(The prime denotes differentiation with respect to R). If we use the Keplerian angular velocity $\Omega = (GM/R^3)^{1/2}$ and the Newtonian viscous torque $T = 2\pi\nu\Sigma R^3\Omega'$, the time evolution of the surface density is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right] \ . \tag{9.11}$$

This is the basic equation for the evolution of a Keplerian accretion disk. Once we know the surface density as a function of time, the radial drift velocity follows from mass conservation:

$$u_R(R,t) = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2}) \ . \tag{9.12}$$

Generally, the kinematic viscosity ν will be a function of R and t. However, the qualitative behavior of a viscous disk can be found by looking at the simplest case, in which $\nu =$ constant. Suppose we start with an accretion disk that is an infinitesimally thin ring, with mass m and radius R_0:

$$\Sigma(R, t=0) = \frac{m\delta(R - R_0)}{2\pi R_0} \ . \tag{9.13}$$

In terms of the dimensionless radius variable $x \equiv R/R_0$ and the dimensionless time variable $\tau \equiv t(12\nu/R_0^2)$, the solution for the surface density is

$$\Sigma(x,\tau) = \frac{m}{\pi R_0^2}\tau^{-1}x^{-1/4}\exp\left[-\frac{1+x^2}{\tau}\right] I_{1/4}\left(\frac{2x}{\tau}\right) \ , \tag{9.14}$$

where $I_{1/4}$ is a modified Bessel function. This solution is plotted in Figure 9.2 at four increasing values of τ. At early times ($\tau \ll 1$), the surface density is nearly Gaussian around $R = R_0$, with a width $\sigma \approx (6\nu t)^{1/2}$. At later times, most of the mass has lost angular momentum and has moved inward, but

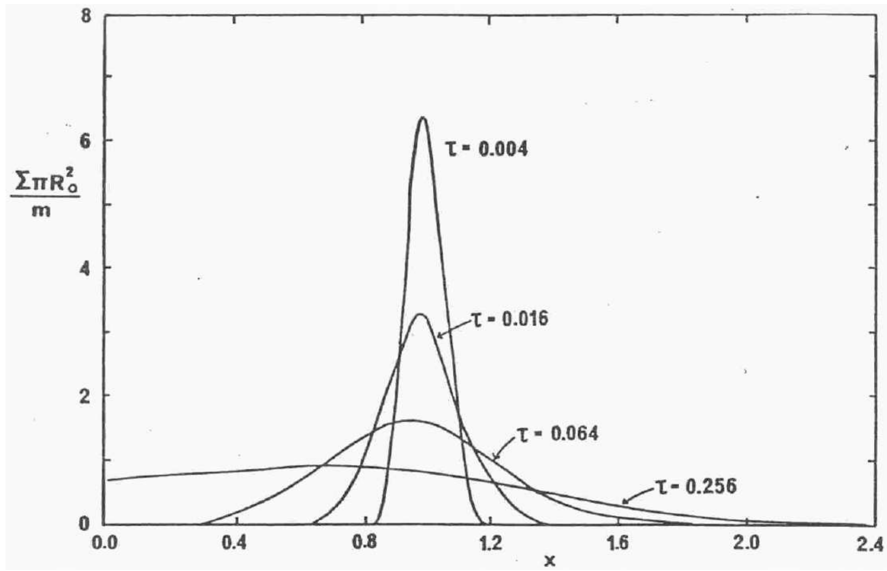

Figure 9.2: The viscous evolution of an initially thin ring of mass m and initial radius R_0. [Pringle, 1981, ARAA, 19, 137]

there is a tail of matter that moves outward to larger and larger radii, and which carries off most of the angular momentum.

A disk that starts as a thin ring will spread with time. In many cases of interest, the accretion disk is fed by a steady stream of matter from outside, and the disk settles into a steady state in which the rate at which gas is lost from the inner edge is equal to the rate at which gas is added at the outer edge. In a steady-state disk of this sort, the radial momentum equation is

$$u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi^2}{R} + \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2} = 0 \ , \tag{9.15}$$

assuming Keplerian rotation. The inflow velocity u_R is of order ν/R, and is much smaller than the rotation velocity u_ϕ. If we define the Mach number of the rotation as $M_\phi \equiv u_\phi/a$,

$$u_\phi = \left(\frac{GM}{R}\right)^{1/2} [1 + O(M_\phi^{-2})] \ . \tag{9.16}$$

Perpendicular to the disk, the gas is in hydrostatic equilibrium, with

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3} \tag{9.17}$$

or

$$\frac{a^2}{\rho}\frac{\partial \rho}{\partial z} \approx -\frac{u_\phi^2}{R^2} z \tag{9.18}$$

for a thin disk. Equation (9.18) can be rewritten in the form

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} \approx -\frac{M_\phi^2}{R^2} z \ . \tag{9.19}$$

This has a solution

$$\rho \approx \rho_0 \exp\left[-\frac{z^2}{2H^2}\right] \ , \tag{9.20}$$

where the scale height is $H \approx R/M_\phi$. Thus, the thin disk approximation ($H \ll R$) is equivalent to saying that the disk rotation is supersonic ($M_\phi \gg 1$).

In a steady-state disk, the inward mass flux is

$$\dot{M} = -2\pi R \Sigma u_R \ . \tag{9.21}$$

Moreover, when $\partial\Sigma/\partial t = 0$, the equation for the conservation of angular momentum can be integrated to yield

$$-\nu\Sigma\frac{\partial \Omega}{\partial R} = -\Sigma u_R \Omega - \frac{C}{2\pi R^3} \ , \tag{9.22}$$

where C is a constant of integration. For a disk in Keplerian rotation around a star of radius R_*, the integration constant is $C = \dot{M}(GMR_*)^{1/2}$. Thus,

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \ . \tag{9.23}$$

When $R \gg R_*$, the surface density of a Keplerian viscous disk will be $\Sigma \approx \dot{M}/(3\pi\nu)$, and the radial velocity will be $u_R \approx -3\nu/(2R)$.

Let $D(R)$ be the rate per unit time per unit area at which the kinetic energy of rotation is dissipated into heat by viscosity. The value of D is

$$D(R) = \frac{1}{2}\nu\Sigma\left(R\frac{\partial \Omega}{\partial R}\right)^2 \ . \tag{9.24}$$

For a Keplerian disk, the viscous dissipation rate is

$$D(R) = \frac{3GM\dot{M}}{4\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \ . \tag{9.25}$$

The formula for the dissipation given above is independent of the value of ν; it merely includes the assumption that the transfer of mass through the accretion disk and the dissipation of energy within the disk are both regulated by the viscosity.

The total disk luminosity is

$$L_{\rm disk} = 2\pi \int_{R_*}^{\infty} D(R) R dR = \frac{1}{2}\frac{GM\dot{M}}{R_*} \ . \qquad (9.26)$$

The disk luminosity is half of the total accretion luminosity $L_{\rm acc} = GM\dot{M}/R_*$. Only half of the luminosity is emitted during the gradual passage of the gas inward through the accretion disk; the other half must be emitted when the gas makes the transition from the inner edge of the accretion disk to the surface of the compact object.

What is the source of the kinematic viscosity ν that causes the inward drift of the gas in the accretion disk? The first guess would be that ν comes from standard molecular viscosity, the result of thermal collisions between individual gas particles in a hot medium. In the case of standard viscosity, the kinematic viscosity is $\nu \sim a_T \lambda$, where where $a_T = (kT/m)^{1/2}$ is a typical thermal velocity, and λ is the mean free path length.

Some more-or-less typical values for accretion disks, at a radius $R \sim 10^{10}\,{\rm cm}$, are $T \sim 10^4\,{\rm K}$ and $n \sim 10^{16}\,{\rm cm}^{-3}$. The mean free path (in an ionized gas) is

$$\lambda = \frac{k^2 T^2}{\pi e^4 n} \sim 10^{-3}\,{\rm cm} \ , \qquad (9.27)$$

the thermal velocity in the accretion disk is $a_T \sim 10^6\,{\rm cm\,s^{-1}}$, and the kinematic viscosity is $\nu \sim 10^3\,{\rm cm^2\,sec^{-1}}$. This value of the kinematic viscosity yields a viscous accretion time scale of $t_{\rm acc} = R^2/\nu \sim 10^{17}\,{\rm sec} \sim 3 \times 10^9\,{\rm yr}$, and the mass will be flowing in at the excruciatingly slow rate of $u_R = -(3\nu)/(2R) \sim -5\,{\rm cm\,yr^{-1}}$.

To explain the observed accretion rates in X-ray binaries and protostars, a much larger value of the kinematic viscosity is required. The Reynolds number for the standard molecular viscosity is ${\rm Re} = u_\phi R/\nu = (GMR)^{1/2}/\nu \sim 10^{15}$ at $R = 10^{10}\,{\rm cm}$ from a $1\,{\rm M}_\odot$ body, with $\nu = 10^3\,{\rm cm^2\,sec^{-1}}$. At such a high Reynolds number, some physicists have argued, we might expect turbulence to set in. The random eddies of the turbulence would cause viscosity, just as random thermal motions cause viscosity on the molecular scale. The kinematic viscosity due to turbulence should be, from dimensional arguments,

$\nu_t \sim \Lambda_t u_t$, where Λ_t is the size of the largest eddies, and u_t is the r.m.s. turbulent velocity. In a thin disk, the largest eddies can be no larger than the disk thickness, so $\lambda_t \lesssim H$. The r.m.s. turbulent velocity is unlikely to be much larger than the sound speed (if it were, shocks would form and the kinetic energy of turbulent motion would become thermalized); thus, we expect $u_t \lesssim a$. These considerations led Shakura and Sunyaev (1973) to invent the **alpha disk**, in which the kinematic viscosity at a given radius is

$$\nu(R, t) = \alpha a(R, t) H(R, t) \ . \tag{9.28}$$

The parameter α is expected, from the arguments given above, to have the numerical value $\alpha \lesssim 1$.

The alpha disk is the most commonly used model for accretion disks. However, by introducing α, we have merely parameterized our ignorance. There is not even agreement as to the actual source of the viscosity. Some astronomers, after pointing out that there are no observations that prove that disks are turbulent, proposed that the viscosity is caused by magnetic stresses due to a tangled magnetic field within the accretion disk. Such a disordered field would give rise to an alpha disk with $\alpha \sim v_A^2/a^2$, where v_A is the Alfven velocity. Other researchers mutter about convective instabilities as a means of transporting angular momentum; others invoke spiral density waves; others propose more exotic mechanisms for transporting angular momentum. In other words, no one knows for sure what causes the viscosity in accretion disks; we just know that it exists, and that the most plausible mechanisms result in viscosities that can be written in the form $\nu = \alpha a H$, where $\alpha \lesssim 1$.