Manipulating Equations

In derivations in class and on the homework assignments, we often need to start with one set of equations and try to end up with another equation, which tells us the quantity we are trying to find or describes some new physical principle. Here are some reminders on the basics of manipulating algebraic equations.

If we start with an equation $a = b$, we can add the same thing to both sides: $a + c = b + c$. Furthermore, if we have a second equation $c = d$, then $a + c = b + d$, since we are still adding the same thing to both sides.

If we have an equation $a = b$, we can multiply both sides of the equation by the same quantity or divide both sides of the equation by the same quantity:

$$ac = bc \qquad \text{or} \qquad a/c = b/c.$$

The one thing you have to be careful of here is not to divide by zero. (I will use ac and $a \times c$ interchangeably to mean "a times c". $a/c = \frac{a}{c}$ means "a divided by c.")

We can multiply fractions together, and we can reverse the order of multiplications ($ab = ba$), so, for example

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{a}{d} \times \frac{c}{b}.$$

Often this kind of rearrangement will allow you to divide numbers that are more convenient to divide (e.g., because they have the same units).

Dividing by a fraction is the same as multiplying by the reciprocal of that fraction (where you switch the numerator and denominator), for example

$$\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

If you have the same quantity in the numerator and denominator, you can cancel it out

$$\frac{ab}{bc} = \frac{a}{c} \qquad \text{and} \qquad a \times \frac{b}{a} = b.$$

You can take the square root of both sides of an equation, or you can raise both sides of an equation to the same power. If $a = b$, then

$$\sqrt{a} = \sqrt{b}, \qquad a^2 = b^2, \qquad a^{3/2} = b^{3/2}.$$

Also

$$\sqrt{a \times b \times c} = \sqrt{a} \times \sqrt{b} \times \sqrt{c},$$

and

$$(a \times b \times c)^2 = (a \times b)^2 \times c^2 = a^2 \times b^2 \times c^2.$$

As an example, suppose we have the equation $x^2/b = a^2$, and we would like to solve for x. Multiplying both sides of the equation by b gives $x^2 = a^2 b$. Taking the square-root of both sides gives $x = a\sqrt{b}$, since the square-root of x^2 is x and the square-root of a^2 is a.

Physical quantities usually come with units, and these units multiply and divide and get raised to powers along with the numbers attached to them. For example, 1 km divided by 1 sec is 1 km/sec, and the distance fallen by an object accelerating at 9.8 m/sec^2 in 10 sec is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(10\mathrm{s})^2 = \frac{1}{2}(9.8)(100)\frac{\mathrm{m}}{\mathrm{s}^2}\mathrm{s}^2 = 490\,\mathrm{m}.$$

If you divide two numbers with the same units, then the units cancel out leaving you with a pure number, e.g., (20 km / 10 km) = 2. Often you can make your life easier by setting up ratios so that units cancel out in this way as much as possible.

Sometimes a homework question will ask you to produce an equation as an answer, and sometimes it will ask you to produce a number with units (e.g., the Schwarzschild radius of the black hole at the center of the Milky Way, or the orbital period of Neptune). In the second case, you want to first get an equation for the quantity you are trying to find, then substitute values to get your number. It is always easiest to work with symbols (M, r, d, a, etc.) for as long as you can, and only substitute numbers at the end; otherwise you do a lot of unnecessary work multiplying numbers and it is easy to make a mistake along the way.

With these rules in mind, let's look back at the calculation in §1 of the course notes.

We started with the general formula for the escape speed:

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}.$$

If $M = M_{\rm earth}$ and $R = R_{\rm earth}$ we get the escape velocity of the earth, and if $M = M_{\rm moon}$ and $R = R_{\rm moon}$ we get the escape velocity of the moon.

The ratio of these two velocities is

$$\frac{v_{\rm esc,moon}}{v_{\rm esc,earth}} = \frac{\sqrt{2GM_{\rm moon}/R_{\rm moon}}}{\sqrt{2GM_{\rm earth}/R_{\rm earth}}} = \sqrt{\frac{2GM_{\rm moon}/R_{\rm moon}}{2GM_{\rm earth}/R_{\rm earth}}} = \sqrt{\frac{M_{\rm moon}/R_{\rm moon}}{M_{\rm earth}/R_{\rm earth}}}.$$

For the second equality we used the fact that the ratio of two square roots is the square root of the ratio: $\sqrt{a}/\sqrt{b} = \sqrt{a/b}$. For the third equality we canceled out 2G between the numerator and denominator.

Now let's use the fact that dividing by c/d is the same as multiplying by d/c (see previous page). Thus

$$\frac{v_{\rm esc,moon}}{v_{\rm esc,earth}} = \sqrt{\frac{M_{\rm moon}}{R_{\rm moon}} \times \frac{R_{\rm earth}}{M_{\rm earth}}} = \sqrt{\frac{M_{\rm moon}}{M_{\rm earth}} \times \frac{R_{\rm earth}}{R_{\rm moon}}}$$

where in the second equality we just reordered the product in the denominator: $R_{\rm moon} \times M_{\rm earth} = M_{\rm earth} \times R_{\rm moon}$.

I told you that the moon is 80 times less massive than the earth ($M_{\rm moon}/M_{\rm earth} = 1/80$) and 4 times smaller in radius ($R_{\rm moon}/R_{\rm earth} = 1/4$), so you can plug in those values to get

$$\frac{v_{\rm esc,moon}}{v_{\rm esc,earth}} = \sqrt{\frac{4}{80}} = \frac{1}{4.47}$$

where the final step requires a calculator (or being good at square roots). Multiplying both sides of this equation by $v_{\rm esc,earth} = 11\,{\rm km\,s^{-1}}$ gives

$$v_{\rm esc,moon} = \frac{v_{\rm esc,earth}}{4.47} = \left(\frac{11}{4.47}\right)\,{\rm km\,s^{-1}} \approx 2.5\,{\rm km\,s^{-1}}$$

where the $\approx$ means "approximately equal to."