9. Stellar Mass Black Holes

Einstein's Field Equation

Einstein's Field Equation, which he discovered (after several years of work) in 1915, describes how matter and energy curve spacetime.

It can be written

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ .$$

This form is deceptively simple. It is really 10 equations, one for each dimension of spacetime and one for each pairwise combination of dimensions.

These are coupled, non-linear, second-order differential equations. Hard to find solutions.

The Field Equation is (roughly) analogous to Newton's equation $F = GMm/r^2$.

A separate (also complicated) equation gives the geodesic path of freely falling objects through the curved spacetime, (roughly) analogous to Newton's $a = F/m$.

Einstein thought that no one would find exact solutions to the Field Equation because it is so complicated.

He calculated the bending of light, gravitational redshift, orbit of Mercury, and properties of gravitational waves using approximate solutions.

Some mathematical detail

In Newton's theory, you need to add up the forces from all masses that are relevant to what you are trying to calculate.

Instead of forces, it is often more convenient to compute the gravitational potential $\Phi(\mathbf{x})$. Acceleration can then be calculated from the gradient of Φ — analogous to sliding around on a rippled or pitted surface.

The equation for the potential can be written

$$\nabla^2\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x})$$

where $\rho(\mathbf{x})$ is the density of mass and ∇^2 is a precise way of saying "changes in Φ" (second derivatives).

Einstein's Field Equation is the generalization of this equation.

The geodesic equation is the analog of $a = -\vec{\nabla}\Phi$ relating acceleration to the gradient of the potential.

See Box 2.6 of Thorne's book for a qualitative description of Einstein's Field Equation.

The Schwarzschild solution

But within three months of Einstein's publishing his Field Equation, Karl Schwarzschild discovered an exact solution describing the curvature of spacetime outside a spherical, non-rotating, star.

Because he assumed all directions are the same and nothing is changing in time, the Field Equation became much simpler, allowing him to find an exact solution.

The properties of the Schwarzschild spacetime can be derived mathematically by looking carefully at his solution to Einstein's equations.

Light traveling on geodesic paths is bent as it passes by the star.

Light going outwards from radius r experiences gravitational redshift.

There is a "critical" radius

$$R_{\rm Sch} = \frac{2GM}{c^2},$$

where M is the mass of the star.

As the distance r approaches $R_{\rm Sch}$:

- The redshift becomes infinite. Light cannot escape from inside $R_{\rm Sch}$.
- The light-bending angle becomes large. Light can travel in a circle at $r = 1.5R_{\rm Sch}$.
- Clocks at r get infinitely out of sync with clocks at large distances.

$R_{\rm Sch}$ is the radius of the "event horizon" – events inside are forever shielded from view.

Plugging in numbers $G = 6.67 \times 10^{-11}{\rm m}^3/({\rm kg\,s}^2)$, $c = 3 \times 10^8$ m/s and $M_\odot = 2 \times 10^{30}\,{\rm kg}$ gives

$$R_{\rm Sch} = 3\,{\rm km} \times \left(\frac{M}{M_\odot}\right) \ .$$

In the *Black Holes and Time Warps* book, Thorne is careful to talk about circumference of the event horizon rather than the radius, because curvature changes the relation between radius and circumference. However, while I concede Thorne's point, I find it more intuitive to talk about radius and just recognize that what we really mean when we say $R_{\rm Sch}$ is the circumference of the event horizon divided by 2π.

Some mathematical detail

Minkowski showed that in Special Relativity all observers agree on the spacetime interval between events,

$$ds^2 = -c^2dt^2 + (dx^2 + dy^2 + dz^2),$$

even though they don't agree on the time and space separations individually.

In spherical coordinates, this could be written

$$ds^2 = -c^2dt^2 + dr^2 + (rd\Omega)^2,$$

where $d\Omega$ is the angular separation (in radians).

Special Relativity describes "flat spacetime" with no gravity, so that geodesic paths are straight lines at constant velocity.

Schwarzschild's solution is a "metric" formula that describes the spacetime interval between nearby events in the curved spacetime around a massive body. It can be written

$$ds^2 = -c^2dt^2\left(1 - \frac{R_{\rm Sch}}{r}\right) + \frac{dr^2}{\left(1 - \frac{R_{\rm Sch}}{r}\right)} + (rd\Omega)^2 \ .$$

From this formula one can calculate $G_{\mu\nu}$, the left-hand side of the Field Equation. The right hand side is the $T_{\mu\nu}$ that corresponds to having a body of mass M at the origin ($r = 0$).

One can also use this formula to calculate the geodesic paths followed by freely falling particles.

What the Schwarzschild Solution Describes

At distances much larger than R_{Sch}, the geodesic paths in Schwarzschild spacetime for objects moving with $v \ll c$ are just like the usual orbits in Newtonian gravity.

The Schwarzschild solution is an accurate description of spacetime curvature in our solar system, to the extent we can ignore the extra curvature caused by the planets themselves.

We don't see the unusual features of Schwarzschild spacetime in our solar system because the sun is much larger than $R_{\text{Sch}} = 3\,\text{km}$; once we are outside the sun, we are already in the regime of "weak gravity."

This is also true throughout the sun, since as we go to smaller radius we also have less interior mass.

The Schwarzschild solution also describes the spacetime around non-spinning black holes.

As we will discuss later, black holes can also have "spin" if they form from the collapse of a rotating star or gas cloud.

The spin of the black hole produces unusual effects on the motions of objects falling towards the black hole.

Stellar mass black holes: theoretical background

1915: Einstein completes the theory of General Relativity

1916: Schwarzschild discovers his solution to Einstein's equations. Einstein and others reject the idea of "Schwarzschild singularities" with event horizons.

1930s:

- Chandrasekhar shows that there is a maximum mass for white dwarfs, stars supported by degenerate electron pressure. A sufficiently massive star cannot end its life as a white dwarf.
- Zwicky suggests the existence of neutron stars, formed in supernovae.
- Oppenheimer and Volkoff show that there is a maximum mass of neutron stars, no more than $3M_\odot$. A sufficiently massive star that has exhausted its nuclear fuel cannot resist collapse to a black hole.
- Oppenheimer and Snyder do calculations of stellar implosion. There are puzzles about the difference seen by observers at large distances and observers moving with the surface of the star ("frozen star" vs. "collapsed star"). These views are eventually resolved (in the 1950s) by Finkelstein, who shows that there is a smooth transition across the event horizon and there is nothing "fishy" about the Schwarzschild solution.

1960s:

- Wheeler coins the term "black hole" (1967)
- Theorists prove that departures from spherical symmetry do not prevent collapse to a black hole.
- Roy Kerr discovers the solution to Einstein's equations that describes the spacetime of spinning black holes (1963).
- Pulsars are discovered, establishing the existence of neutron stars. Some are found in supernova remnants, establishing the link to stellar collapse and supernova explosions.

By 1969 there is lots of theoretical evidence that the collapse of massive stars at the ends of their lives should be able to produce black holes.

There is a reasonably good understanding of what the properties of black holes should be.

But there is still no direct empirical evidence for the existence of any stellar mass black holes.

Finding Black Holes

If all stars with initial mass $M > 30M_{\odot}$ form black holes after they use up their fuel and collapse, then we expect something like 100 million (!) black holes in the Milky Way (compared to 100 billion stars in total).

If we suspect black holes exist, how should we search for them?

Even the nearest would be much too far away to see a "black dot."

Gas falling onto a black hole would heat up as it runs into itself. Temperatures near event horizon would be high, producing X-ray emission.

Where can gas come from? Diffuse interstellar gas too thin to produce much glow.

But roughly half of all stars are found in binary systems, with an orbiting companion.

When the orbiting companion of the black hole expands to become a red giant, part of its envelope can get pulled over onto the black hole.

The same thing can happen for neutron stars.

X-ray Astronomy

X-rays are absorbed in the earth's atmosphere.

Can't see X-rays from astronomical sources from ground.

But stars aren't that hot, so in early 1960s no one expected to see many bright astronomical X-ray sources.

First X-ray experiments, launched on sub-orbital rockets, showed some very strong sources.

First X-ray satellite, *Uhuru* (1970), showed many X-ray sources.

Later satellites improved sensitivity and angular resolution of X-ray images. Two of the most important: *Einstein* (1978) and *Chandra* (1999, still active).

One main population of X-ray sources: neutron stars and black holes accreting gas from binary companions.

(Another main population, supermassive black holes in distant galaxies, will come up later).

There are now about 400 X-ray binaries known in our own Galaxy (the Milky Way). Probably $\frac{1}{4} - \frac{1}{3}$ are black holes; the rest are neutron stars.

These are a tiny fraction of the 100 million black holes we expect to be present in our galaxy, but the conditions for a black hole to "light up" are rare, and they don't last long.

We can also detect X-ray binaries in nearby galaxies.

Demonstrating that an X-ray Source is a Black Hole

How do you show that an X-ray source is an accreting black hole?

Need to show that its mass is larger than the largest possible mass of a neutron star.

How do we measure the mass of something we can't see?

Measure its gravitational effect on something we do see.

If a star is orbiting around another object, the lines in its spectrum (see §8) shift back and forth periodically because of Doppler shifts.

Amount of shift gives velocity.

Period and velocity give acceleration, hence mass of object causing the orbit. Roughly like applying Kepler's 3rd law to get mass of central object.

For a black hole candidate, need to find the visible-light companion of the X-ray source.

This was hard because of poor resolution of early X-ray telescopes: many possible companions.

Once identified, measure motion of companion by Doppler shifts. Infer mass of X-ray source.

If this is above $3M_\odot$, then the object can't be a neutron star: must be a black hole.

Accretion Disks

Gas coming from companion does not fall straight onto the black hole, because companion is moving.

Gas goes into orbit around black hole, forming a disk.

"Friction" causes some gas to slide inwards towards the BH.

This "friction" also heats the gas up.

(In reality, the friction is probably a complicated interaction involving magnetic fields and moving gas).

We see X-rays (and ultraviolet and visible light) from hot gas in the accretion disk.

In an accretion disk, gas gains energy from gravity as it "sinks" towards the black hole. Some of that energy goes into motion and heat of the gas, but some of it is carried away by radiation (X-rays, etc.), making the accreting black hole luminous.

In some cases, gas is ejected at high speed along jets perpendicular to the disk, producing X-ray, visible light, and radio emission.

It's hard to measure gas motions very close to the black hole because that gas is emitting X-rays, not visible light.

Current X-ray telescopes can measure spectral lines from highly ionized iron atoms close to the black hole.

The shapes of the iron lines show strong Doppler shifts, both red shift and blue shift, indicating gas that is orbiting at a significant fraction of the speed of light.

(I'm using the terms "red" and "blue," but I'm really talking about the wavelengths of X-ray photons getting longer or shorter, not visible light.)

The light is also gravitationally redshifted as it climbs away from the strong gravity of the black hole.

Since Doppler shifts make some light bluer and some light redder (whether its coming toward us or away from us), but gravitational redshift only makes the light redder, more light is redshifted than blueshifted.

Event Horizons

In General Relativity, the fundamental feature of a black hole is an event horizon.

Since we are not close to any black holes, how can we tell whether these event horizons actually exist?

Suppose the flow of gas from the companion slows to a trickle.

The X-ray luminosity goes down. But if the companion is a neutron star, there will always be some emission when the gas strikes the surface of the neutron star and heats up.

For a black hole, the trickle could disappear down the event horizon without emitting much energy at all.

X-ray binaries vary in brightness, probably because of variations in the gas supply.

If you separate X-ray binary sources with estimated masses into those where the companion appears to be a neutron star ($M < 2M_\odot$) and those where the companion appears to be a black hole ($M > 3M_\odot$) based on mass, the *minimum* luminosity is much lower for the black holes.

This suggests that infalling gas really is disappearing down the event horizon of a black hole, not hitting the surface of a compact object.

While we have strong evidence for the existence of compact X-ray sources that are too massive to be neutron stars, this indirect argument is the best empirical evidence we have that these objects actually have event horizons.

Summary: Empirical Evidence for Stellar Mass Black Holes

We have previously reviewed the theoretical arguments that black holes *could* exist and *could* form from the core collapse of massive stars that have exhausted their nuclear fuel.

We can now characterize the chain of direct observational evidence for the existence of stellar mass black holes.

- X-ray telescopes reveal bright sources in the Milky Way and nearby galaxies, apparently gas falling onto compact objects from (normal) stellar companions.

- Some of these sources are neutron stars.
- But velocity measurements show that *some* of the compact objects have masses above $3M_{\odot}$. According to all we know about neutron stars and gravity, these can only be completely collapsed objects – black holes.
- With today's X-ray telescopes, we can measure the detailed spectrum of the X-ray emission (intensity of X-rays as a function of energy) of some sources, and find agreement with expectations for gas accreting onto black holes.
- For example, the X-ray lines from iron atoms show large Doppler shifts (gas moving close to the speed of light) and gravitational redshifts.
- Some sources show jets of material (seen in X-rays, visible light, or radio waves).
- Low minimum luminosities of X-ray sources believed (based on mass) to be black holes suggests that these source really do have *event horizons* rather than hard surfaces.

More recently (2015 and after), gravitational waves provide very convincing evidence for stellar mass black holes with the properties predicted by GR. But we'll come to that later.