Highlight Calculations From Astronomy 2142

Eugene Wigner's essay emphasizes the extraordinary success of mathematical calculations in deriving conclusions about the natural world. In this course, we have stuck to calculations that are either idealized or approximate (or both) so that we can do them with reasonably simple algebra. Nonetheless, we have been through a number of impressive examples of using calculations to draw powerful conclusions, in lecture, in class questions, and in homework assignments.

Here are the calculations that I think of as Astronomy 2142's highlight examples of applying mathematical reasoning to reach physical conclusions.

1. Newton's inverse-square law of gravity explains Kepler's 3rd law of planetary motion, for circular orbits.

Equations used:

$$F = \frac{GMm}{r^2}\ , \qquad a = \frac{F}{m}\ , \qquad a = \frac{v^2}{r}$$

Course section 3.

2. Derivation of the mass ratio $M_{\rm Jup}/M_{\rm Sun}$.

Done with the same equations, plus the orbital periods of Earth and Ganymede.

Homework #1.

3. Momentum conservation and equivalence of reference frames implies $E = mc^2$.

Inspired by Einstein (1905), we considered a box emitting radiation as seen from two different reference frames, and concluded that the inertial mass of the box decreased by E/c^2.

Course section 4.

4. Derivation of the formula for time dilation:

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

We used $l = ct$ plus the geometry of a moving light clock and the Pythagorean theorem, with the assumption that the speed of light is the same in the stationary and moving frames.

Course section 5.

5. The possible lifetime of the sun if it is powered by (a) chemical burning or (b) hydrogen fusion

Conclusion: Chemical burning can power the sun for a few thousand years, but fusion can power it for 10 billion years.

Homework # 2.

6. The impact of special relativistic time dilation on clocks carried by airplanes

Approximate agreement (in the magnitude of time difference) with the results of the Hafele-Keating experiment.

Homework #2.

7. Gravitational redshift of EM radiation:

$$\lambda_o = \lambda_e \times \left(1 + \frac{gL}{c^2}\right)$$

We used the equivalence principle and the Doppler formula $\lambda_o = \lambda_e \times (1 + v/c)$.

By further considering wave pulses emitted and received at the bottom and top of a box, we concluded that clocks deeper in a gravitational field run more slowly.

Course section 6.

8. The "gravitational mass" of a photon of energy E is E/c^2.

We used the gravitational redshift formula and the formula mgL for gravitational potential energy of a body of mass m.

Class question during course section 6.

9. The energy of a core collapse supernova

You used the formula $E = GM^2/R$ for gravitational potential energy released by forming a neutron star of mass M and radius R and compared that to the energy $E_\odot = 0.001 M_\odot c^2$ that the Sun releases from hydrogen fusion during its 10 billion year lifetime, finding that the former exceeds the latter by a factor of 300.

Homework #3.

10. The shape of X-ray emission lines from ionized iron in an accretion disk

Your answer here was only qualitative, but you used the Doppler formula and gravitational redshift formula to find the width and asymmetry of the 6.4 keV iron line from the inner regions of an accretion disk, explaining the basic features observed by X-ray telescopes.

Homework #3.

11. The luminosity of an accreting black hole compared to a galaxy and the growth rate of a black hole shining close to the Eddington luminosity

You used the equations

$$L = \frac{1}{12}\dot{M}c^2 , \qquad L_{\mathrm{Edd}} = 3.4 \times 10^5 L_\odot \left(\frac{M}{M_\odot}\right)$$

to find that an accreting supermassive black hole could outshine an entire galaxy of stars and that a black hole shining near the Eddington luminosity would roughly double its mass in about 50 million years.

Class question in course section 11 and Homework #4.

12. The required sensitivity of LIGO

We used a guess at the rate of stellar black hole mergers per galaxy and the approximate equation

$$h \approx \frac{1}{5}\frac{R_{\mathrm{Sch}}}{D}$$

plus the definition $R_{\mathrm{Sch}} = 2GM/c^2 = 3\,\mathrm{km}(M/M_\odot)$, concluding that LIGO would need a sensitivity of about $h \approx 3 \times 10^{-22}$ to detect at least one BH merger per year.

Course section 12.

13. The distortion of the earth from a merger of $10^6 M_\odot$ BHs at the center of the Milky Way

Equations used:

$$h \approx \frac{1}{5}\frac{R_{\mathrm{Sch}}}{D} , \qquad h = \frac{\Delta d}{d}$$

Conclusion: Earth's diameter would oscillate by 0.025 millimeters.

Class question in course section 12.

14. The BH masses and distance of the event LIGO GW150914, from the measured waveform

Equations used:

$$P \approx 2\pi R_{\rm Sch}/c\ , \qquad h \approx \frac{1}{5}\frac{R_{\rm Sch}}{D}\ , \qquad h = \frac{\Delta d}{d}$$

Homework #4.

15. Ability of the Event Horizon Telescope to resolve the Milky Way and M87 BHs

Equations used:

$$\theta = \frac{l}{D}\ , \qquad \theta_{\rm min} = \frac{\lambda}{D}$$

(Remember that in the first equation D refers to the distance of the object, and in the second equation D refers to the distance between telescopes.)

Class question in course section 13.

16. Evaporation time of a BH via Hawking radiation

Equations used:

$$E = Mc^2\ , \qquad E_\gamma = \frac{hc}{\lambda}\ , \qquad \lambda \approx R_{\rm Sch}\ , \qquad R_{\rm Sch} = \frac{2GM}{c^2}$$

Class question in course section 14.

If I were to pick the highlights of these highlights, I would choose numbers 1, 3, 4, 7, 8, 14, and 16.