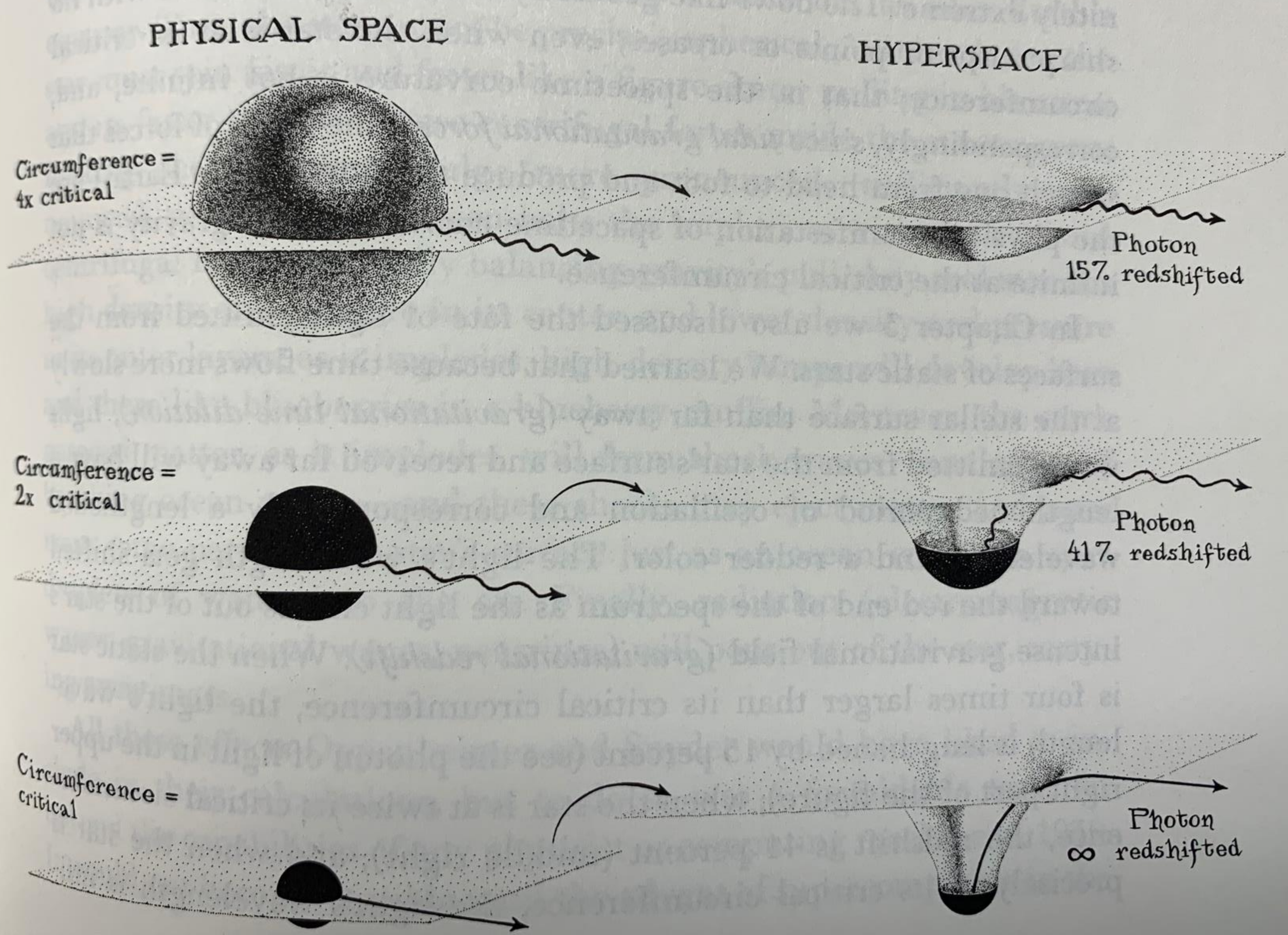
$$ds^2 = \left(1 - \frac{2Gm}{c^2 r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2Gm}{c^2 r}\right)} dr^2 - (r)^2 (d\theta^2 + \sin^2(\theta) d\phi^2)$$

The Schwarzschild spacetime metric

Karl Schwarzschild

6.2 (Same as Figure 3.4.) General relativity's predictions for the curvature of space and the redshift of light from a sequence of three highly compact, static (non-imploding) stars that all have the same mass but have different circumferences.



David Weinberg
11/17/83

10

18.6 — Bending of Light by the Sun

From problem 16.2, the geodesic equation for the photon is $\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma = (\nabla_{\vec{p}} \vec{p})^\alpha = 0$, where $\vec{p} = \frac{d}{d\lambda}$ = 4-momentum of photon = tangent vector to world line of photon. In a weak gravitational field, the photon moves along this geodesic, a slight deflection from the world line $x=t$, $y=b$, $z=0$.

To evaluate $\frac{dp^y}{d\lambda}$, we need the connection coefficients for the metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)(dx^2+dy^2+dz^2) \qquad r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{dp^y}{d\lambda} = -\Gamma^y_{\alpha\beta} p^\alpha p^\beta = -\left(\Gamma^y_{00} p^0 p^0 + 2\Gamma^y_{x0} p^x p^0 + \Gamma^y_{xx} p^x p^x\right)$$
(since $p^y \approx p^z = 0$).

To evaluate the connection coefficients, we note that

$$g_{00,x} = g_{xx,x} = g_{yy,x} = -\frac{1}{2}\cdot(2x)\cdot 2M(x^2+y^2+z^2)^{-3/2} = -\frac{2Mx}{r^3}$$

$$g_{00,y} = g_{xx,y} = g_{yy,y} = \frac{-2My}{r^3} \qquad g_{00,0} = g_{xx,0} = g_{yy,0} = 0$$

and

$$\Gamma^y_{00} = \frac{1}{2}\cdot\eta^{y\mu}(g_{\mu 0,0} + g_{\mu 0,0} - g_{00,\mu}) = \frac{1}{2}(-g_{00,y}) = \frac{My}{r^3}$$

$$\Gamma^y_{x0} = \frac{1}{2}\cdot(g_{yx,0} + g_{y0,x} - g_{x0,y}) = 0$$

$$\Gamma^y_{xx} = \frac{1}{2}(g_{yx,x} + g_{xy,x} - g_{xx,y}) = \frac{My}{r^3}$$

recalling that, in linearized theory, indices are raised and lowered with the Minkowski metric. The geodesic equation therefore yields

$$\frac{dp^y}{d\lambda} = -(\Gamma^y_{00} p^0 p^0 + \Gamma^y_{xx} p^x p^x) = -\frac{My}{r^3}(p^0 p^0 + p^x p^x)$$

$$\cong -\frac{2My}{r^3} p^x p^x \qquad (\text{since } p^0 \approx p^x) ✓$$

$$= \frac{-2My}{r^3} p^x \frac{dx}{d\lambda} \qquad \left(\text{since } p^x = \frac{\partial}{\partial x}\cdot\vec{p} = \frac{\partial}{\partial x}\cdot\left(\frac{d}{d\lambda}\right) = \frac{\partial}{\partial x}\cdot\left(\frac{\partial}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\right) = \frac{dx}{d\lambda}\right)$$

$$\cong \frac{-2Mb}{(x^2+b^2)^{3/2}} p^x \frac{dx}{d\lambda} \qquad (\text{approximating } y=b \text{ throughout}) ✓$$

18.6 – cont.

Since the photon's path must be a null geodesic, $\vec{p}\cdot\vec{p} = 0$. If $|p_y| \ll p^0 \approx p^x$, this implies that

$$\vec{p}\cdot\vec{p} \cong g_{00} p^0 p^0 + g_{xx} p^x p^x = 0$$

$$\Rightarrow (p^x)^2\left(1+\frac{2M}{r}\right) = (p^0)^2\left(1-\frac{2M}{r}\right)$$

$$p^x = p^0\sqrt{\frac{1+2M/r}{1-2M/r}} = p^0\sqrt{\frac{1-4M^2/r^2}{1-4M/r+4M^2/r^2}} ✓$$

The minimum value of r is b, so approximating to first order in $\frac{M}{b}$ gives a maximum deviation (ignoring $|p_y|$) of

$$p^x = p^0(1-4M/b)^{-1/2} \cong p^0(1+2M/b). ✓$$

As shown in problem 18.7, p_0 = constant of motion ✓ $\Rightarrow p^0 = \eta^{00}p_0$ = constant of motion (indices raised with $\eta_{\mu\nu}$ in linearized theory). Thus $p^x = \text{const.}(1+\mathcal{O}(M/b))$. For the bending of light by the sun, $b > R_\odot \Rightarrow \frac{M}{b} \ll 1$, so p^x can be treated as essentially constant over the photon's path.

This allows a solution for $p^y(x=+\infty)$:

$$\frac{dp^y}{d\lambda} = \frac{-2Mb}{(x^2+b^2)^{3/2}} p^x \frac{dx}{d\lambda} \Rightarrow dp^y = -2Mb\, p^x (x^2+b^2)^{-3/2} dx$$

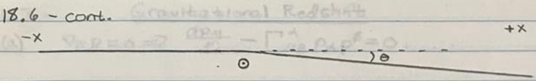
$$\Rightarrow \int_{-\infty}^{\infty} dp^y = -2Mbp^x \cdot 2\int_0^\infty (x^2+b^2)^{-3/2} dx ✓$$

$$p_y(x=+\infty) - p_y(x=-\infty) = -4Mbp_x \left(\frac{x}{b^2\sqrt{x^2+b^2}}\right)\Big|_0^\infty$$

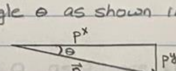
$$p^y(x=+\infty) = -4Mbp_x\cdot\left(\frac{1}{b^2}\right) = \frac{-4M}{b}p^x ✓$$

For $p^y(x=-\infty) = 0$

18.6 – cont.



The photon is deflected by an angle θ as shown in the figure where $\sin\theta \cong \theta = \left|\frac{p_y}{p_x}\right|$.



As shown already $\left|\frac{p_y}{p_x}\right| = \frac{4M}{b}$

$$\text{So} \quad \theta = \frac{4M}{b} = \frac{4}{b}\left(R_\odot\cdot\frac{M_\odot}{R_\odot}\right) = \frac{4R_\odot}{b}\cdot\left(\frac{1.48\times10^5\text{ cm}}{6.96\times10^{10}\text{ cm}}\right)$$

$$= 8.5\times10^{-6}\left(\frac{R_\odot}{b}\right)\text{ radians} = 1.75''\left(\frac{R_\odot}{b}\right) ✓$$

11/17/83

10

18.6 — Bending of Light by the Sun

From problem 16.2, the geodesic equation for the photon is $\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma = (\nabla_{\vec{p}}\vec{p})^\alpha = 0$, where $\vec{p} = \frac{d}{d\lambda}$ = 4-momentu of photon = tangent vector to world line of photon. In a weak gravitational field, the photon moves along this geodesi a slight deflection from the world line $x=t$, $y=b$, $z=0$.

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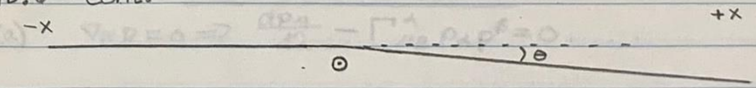
$$\frac{dp^y}{d\lambda} = -\Gamma^y_{\alpha\beta}p^\alpha p^\beta = -\left(\Gamma^y_{00}p^0p^0 + 2\Gamma^y_{x0}p^xp^0 + \Gamma^y_{xx}p^xp^x\right)$$

(since $p^y \approx p^z = 0$).

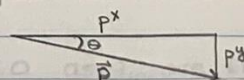
To evaluate the connection

$$g_{00,x} = g_{xx,x} = g_{yy,x} = -\tfrac{1}{2}\cdot(2x)\cdot 2M$$

18.6 - cont.



The photon is deflected by an angle θ as shown in the figure where $\sin\theta \cong \theta = \left|\frac{p^y}{p^x}\right|$.



As shown already $\left|\frac{p^y}{p^x}\right| = \frac{4M}{b}$

So $\theta = \frac{4M}{b} = \frac{4}{b}\left(R_\odot \cdot \frac{M_\odot}{R_\odot}\right) = \frac{4R_\odot}{b}\cdot\left(\frac{1.48\times10^5\ \text{cm}}{6.96\times10^{10}\ \text{cm}}\right)$

$= 8.5\times10^{-6}\left(\frac{R_\odot}{b}\right)$ radians $= 1.75''\left(\frac{R_\odot}{b}\right)$ ✓

Theory of black holes: brief history

- 1915: Einstein completes theory of General Relativity (GR).
- 1916: Schwarzschild discovers exact solution. Einstein and others reject the idea of "Schwarzschild singularities" with event horizons.

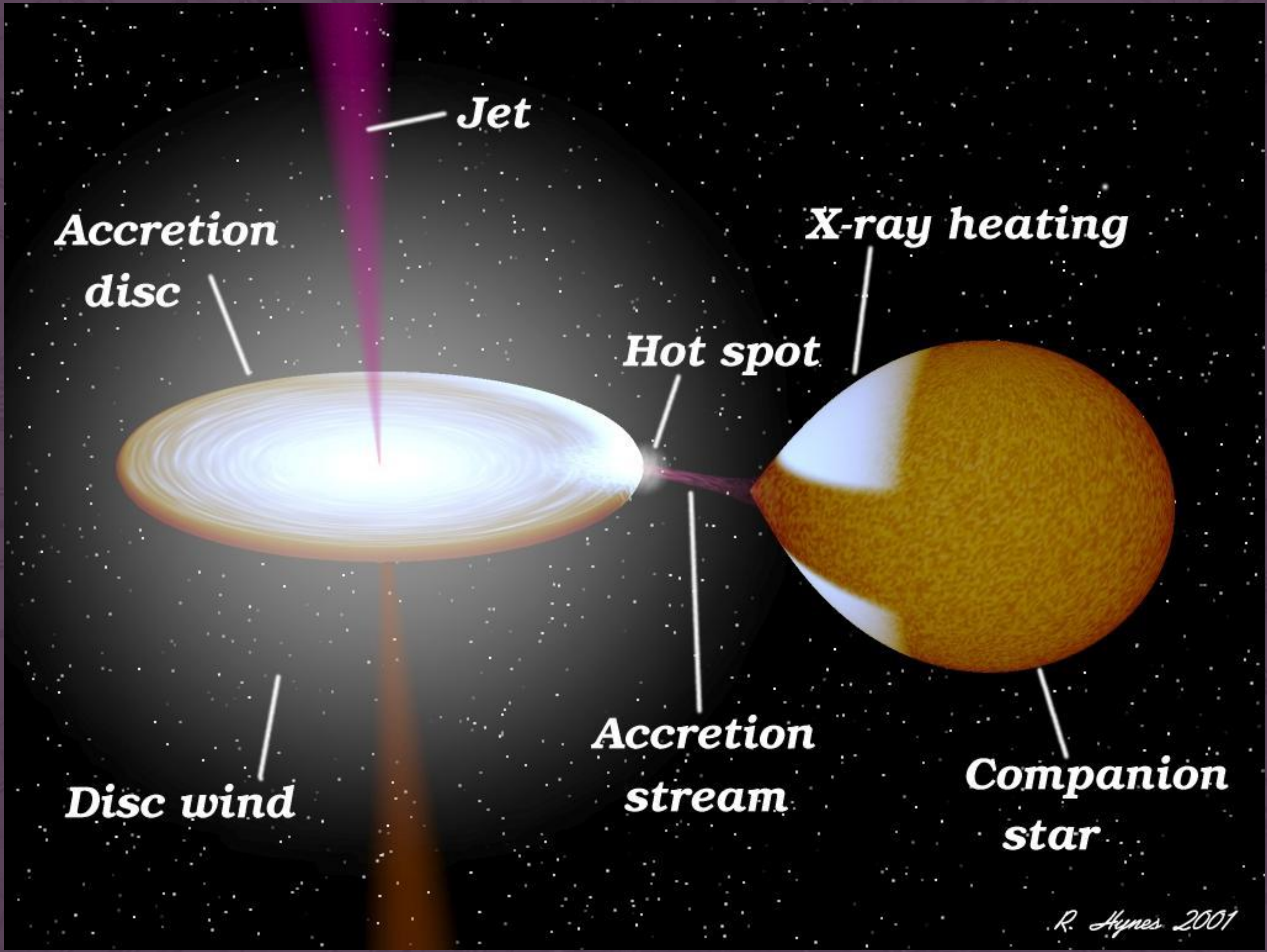
1930s:

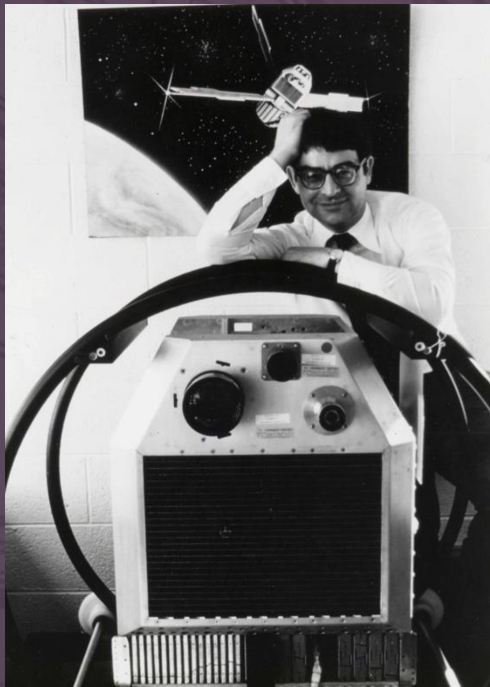
- Chandrasekhar -- maximum mass of white dwarfs.
- Zwicky – neutron stars, formed in supernovae
- Oppenheimer & Volkoff – maximum mass of neutron stars
- Oppenheimer & Snyder – calculations of stellar implosion. "Frozen star" or "collapsed star" depending on reference frame.

1960s

- Finkelstein (1958) – "Non-fishy" description of Schwarzschild spacetime.
- Wheeler (1967) – coins the term "black hole".
- Various – Proof that departures from spherical symmetry don't prevent collapse.
- Kerr (1963) – New exact solution to Einstein's equations, describes the spacetime around spinning black holes.

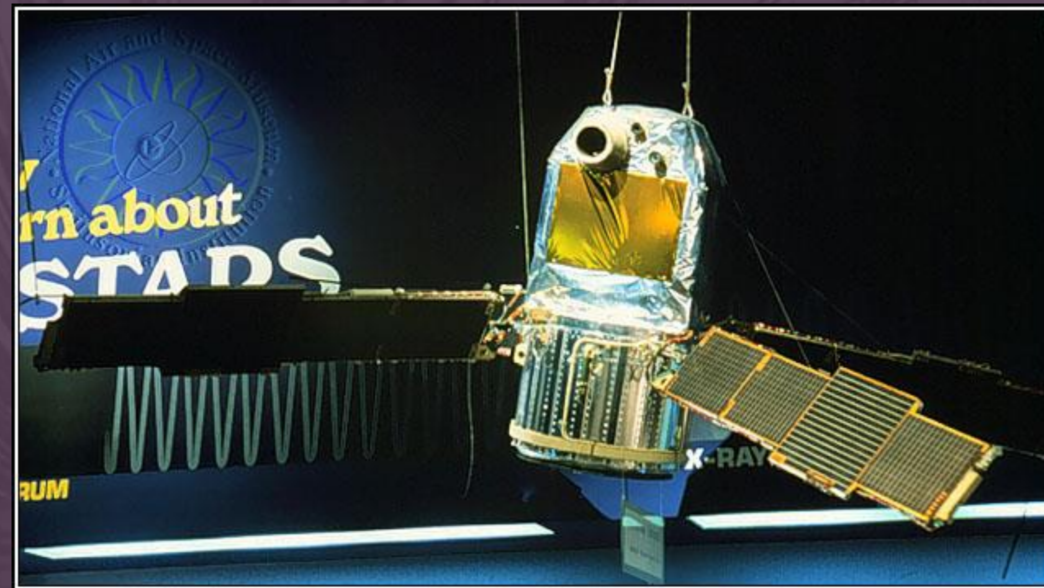
Pulsars (spinning neutron stars) are discovered in 1967.

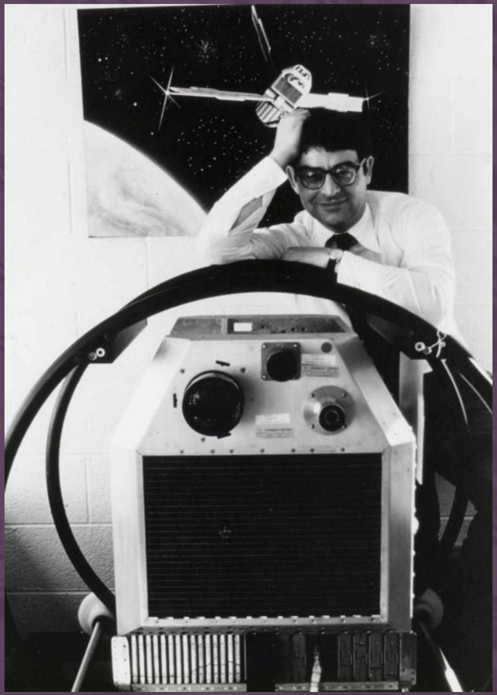

Jet
Accretion disc
X-ray heating
Hot spot
Accretion stream
Companion star
Disc wind
R. Hynes 2001



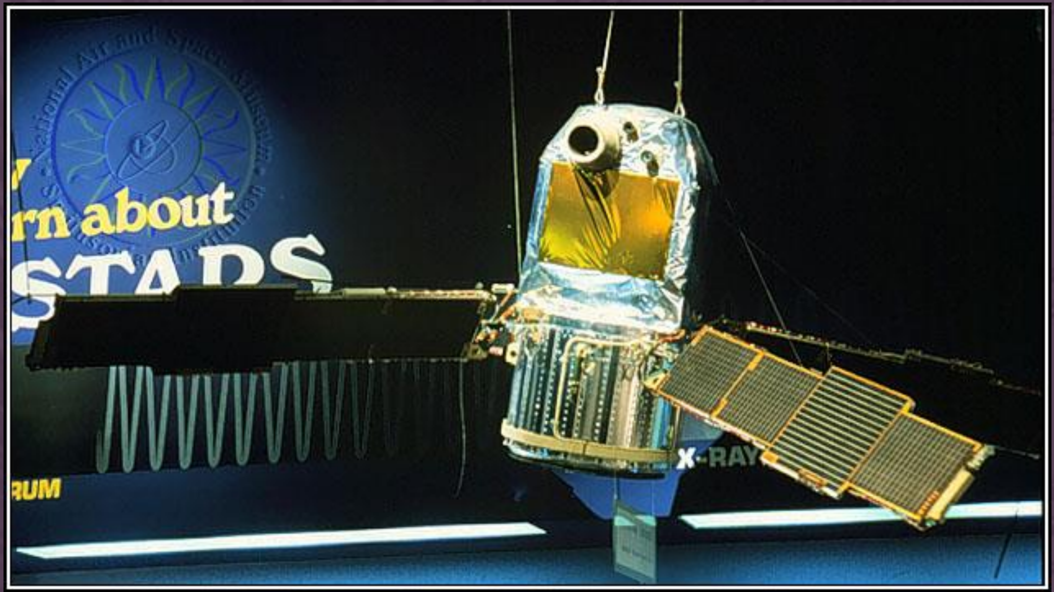
Ricardo Giacconi – Lead scientist of Uhuru, winner of 2002 Nobel Prize in Physics

Uhuru satellite, first X-ray telescope

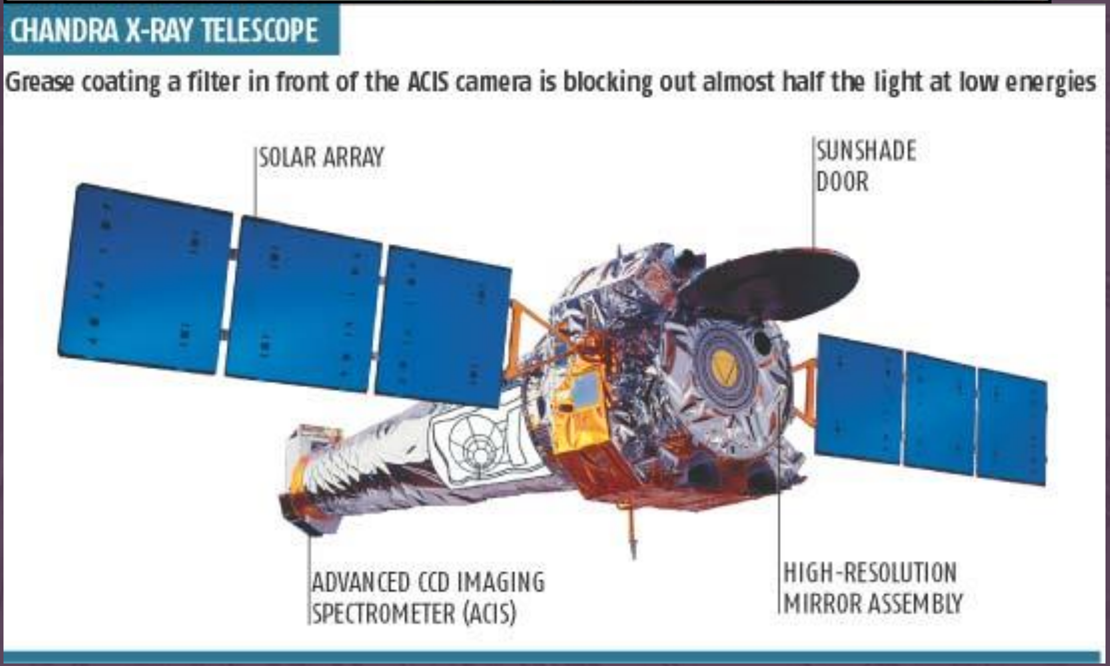




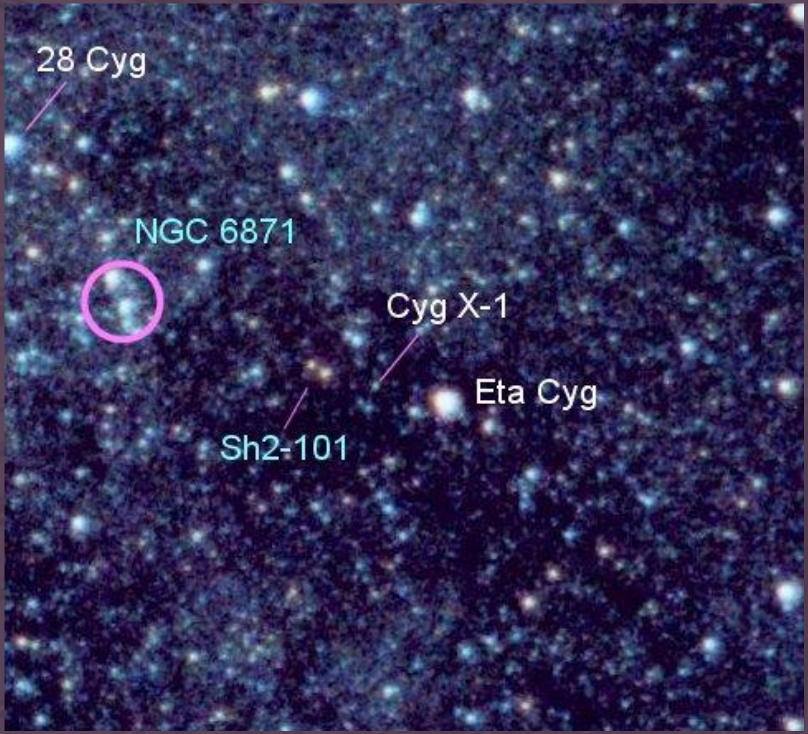
Uhuru satellite, first X-ray telescope



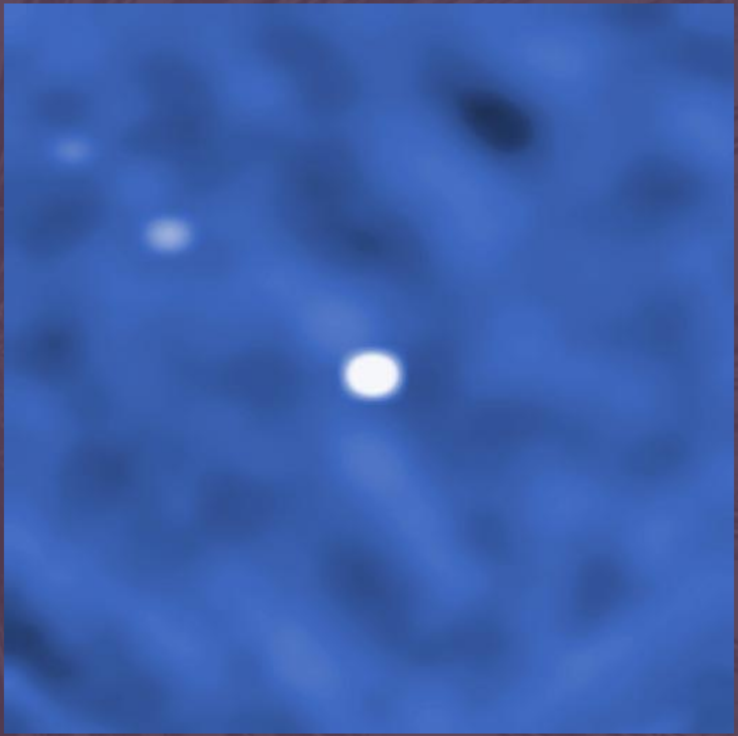
Nested “mirrors” of Chandra X-ray telescope



Optical light



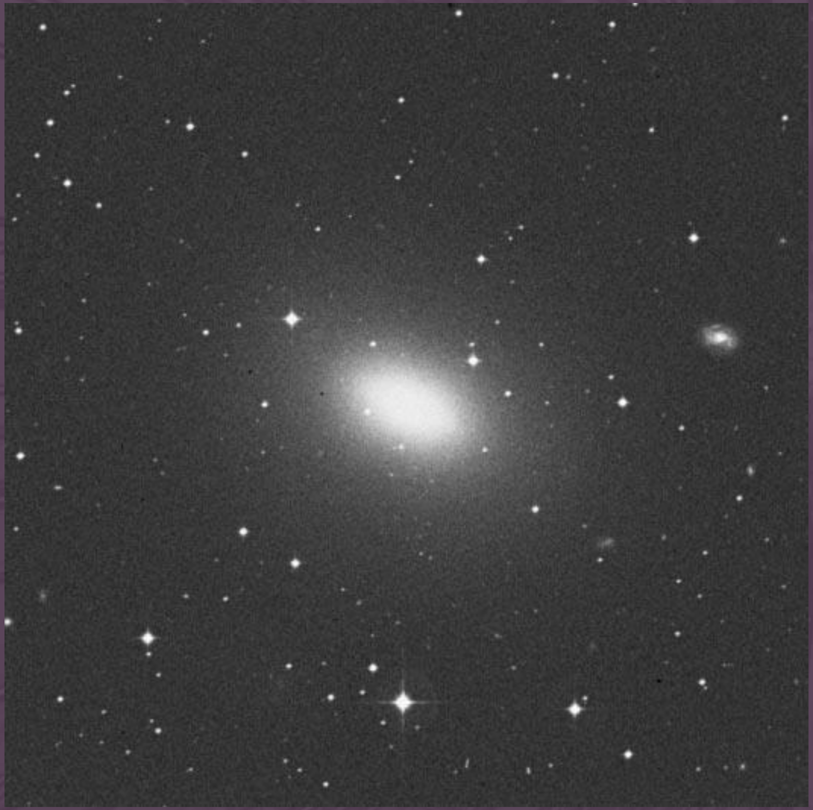
Gamma rays



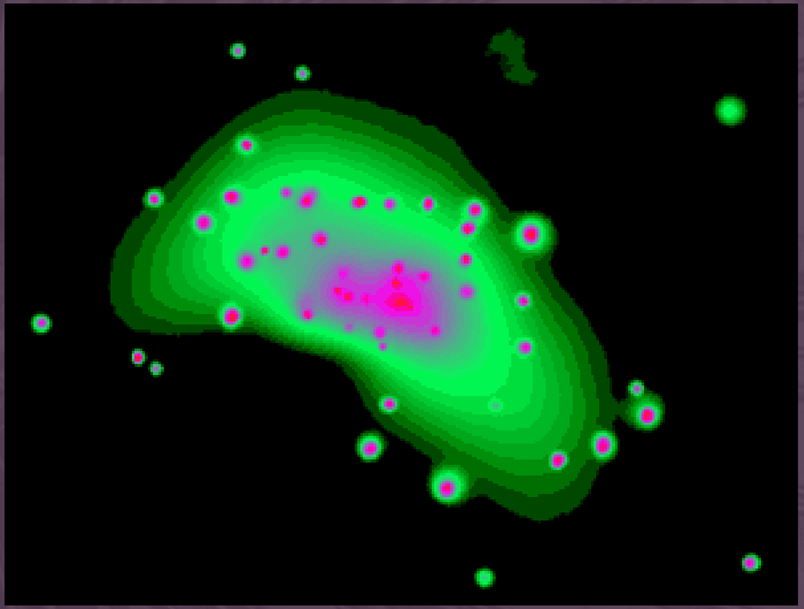
(INTEGRAL satellite, neutron star Cyg X-3 at upper left)

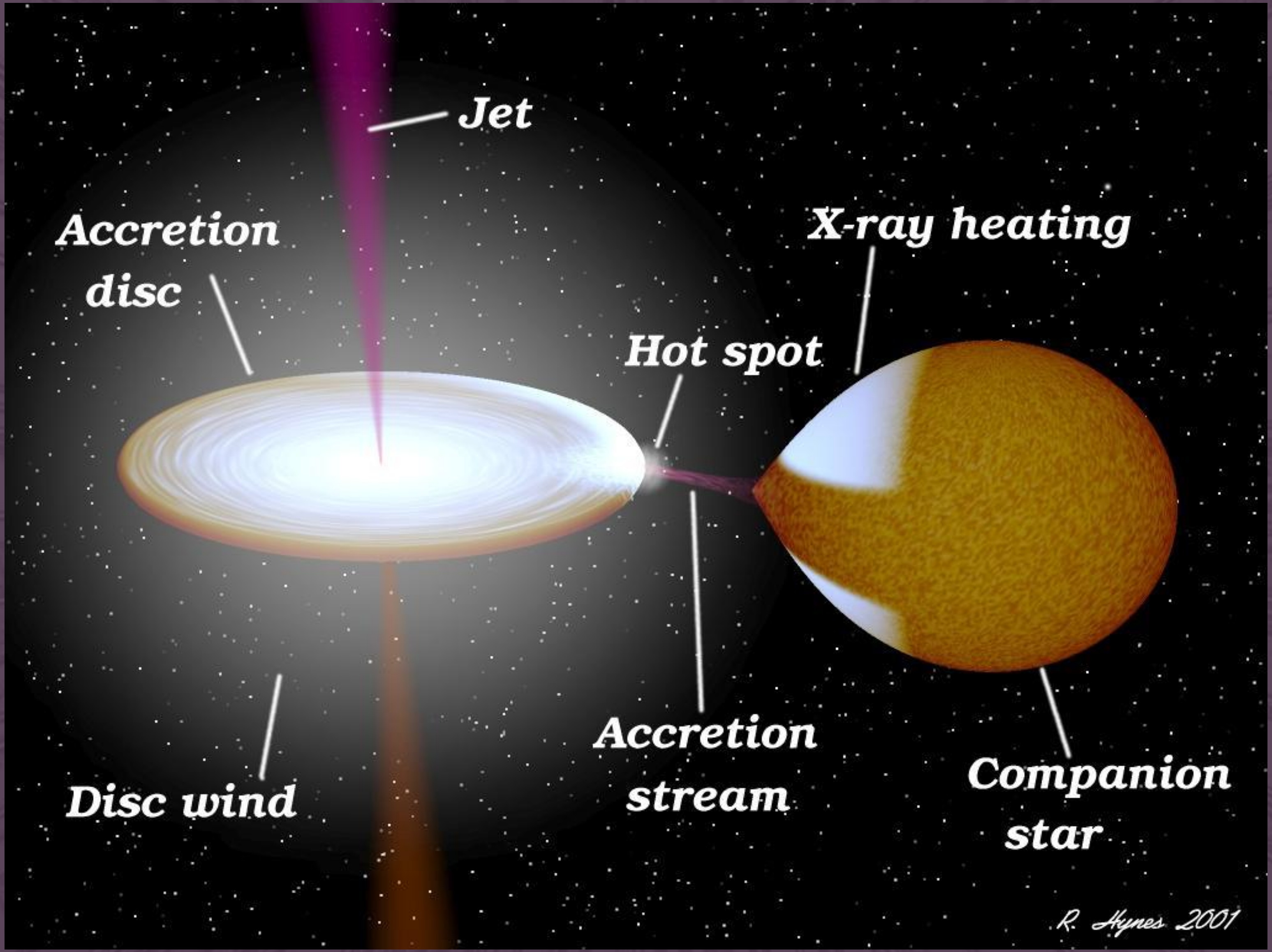
Galaxy NGC 4697

Optical light



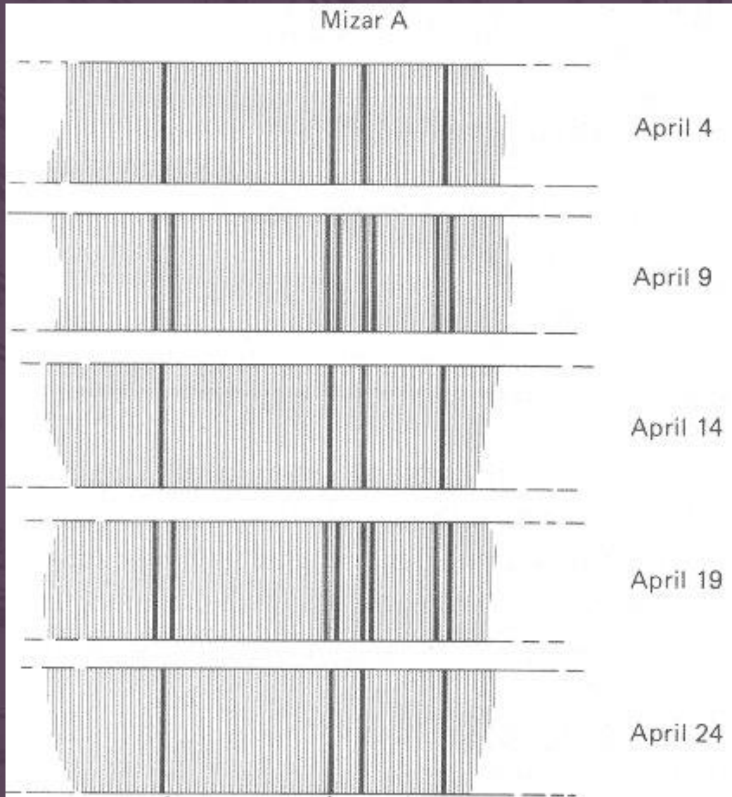
X-rays

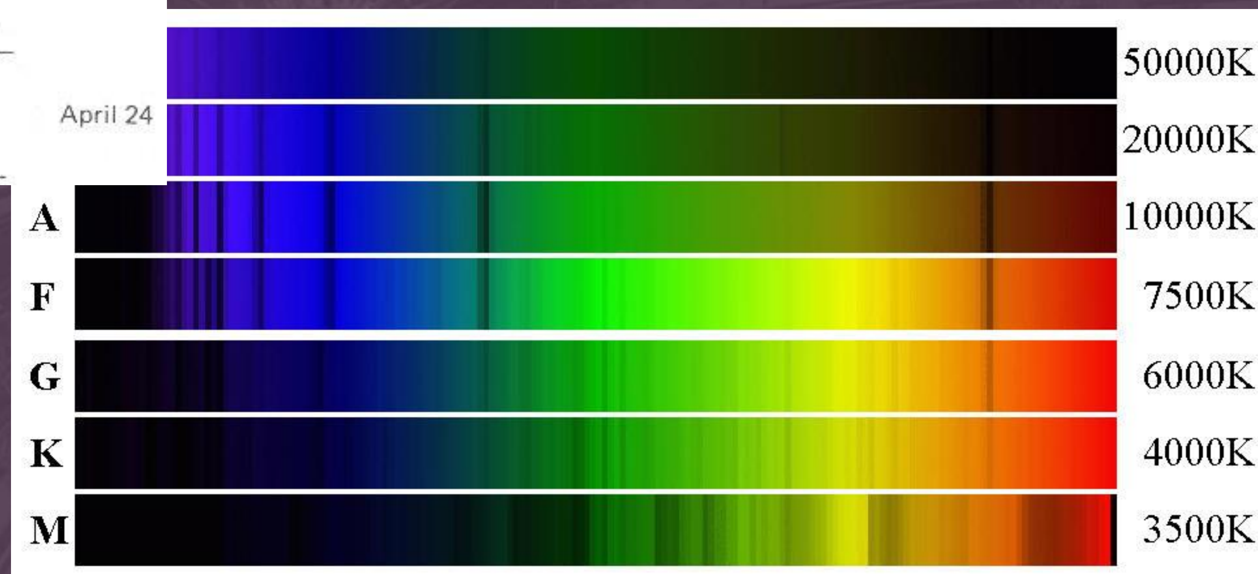




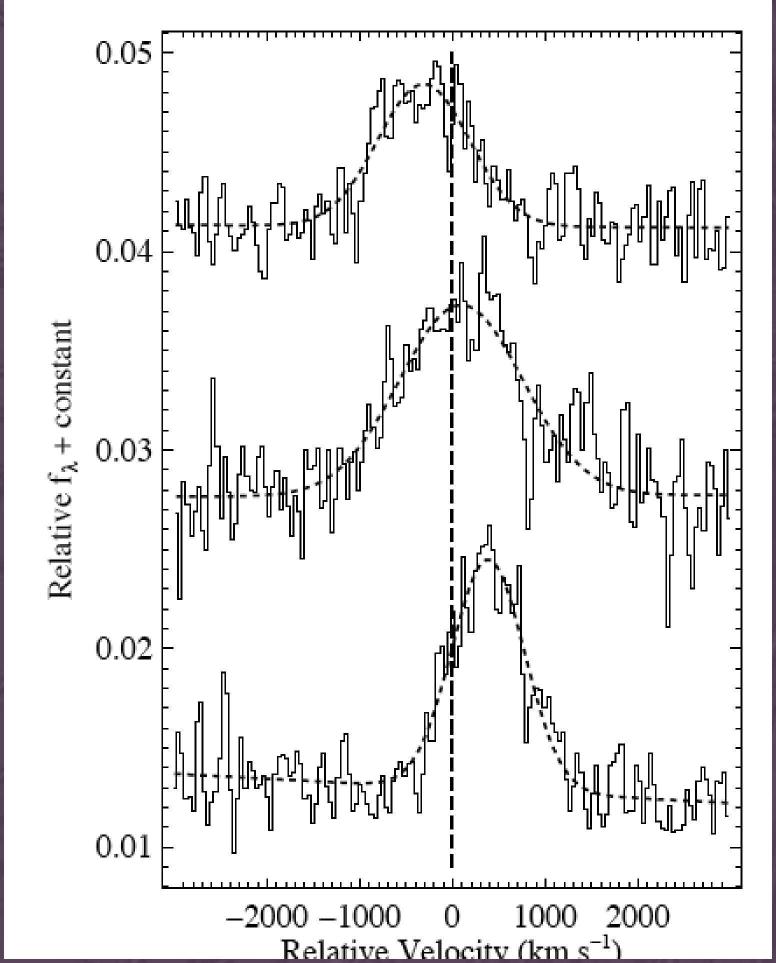
Jet
Accretion disc
X-ray heating
Hot spot
Accretion stream
Disc wind
Companion star
R. Hynes 2001

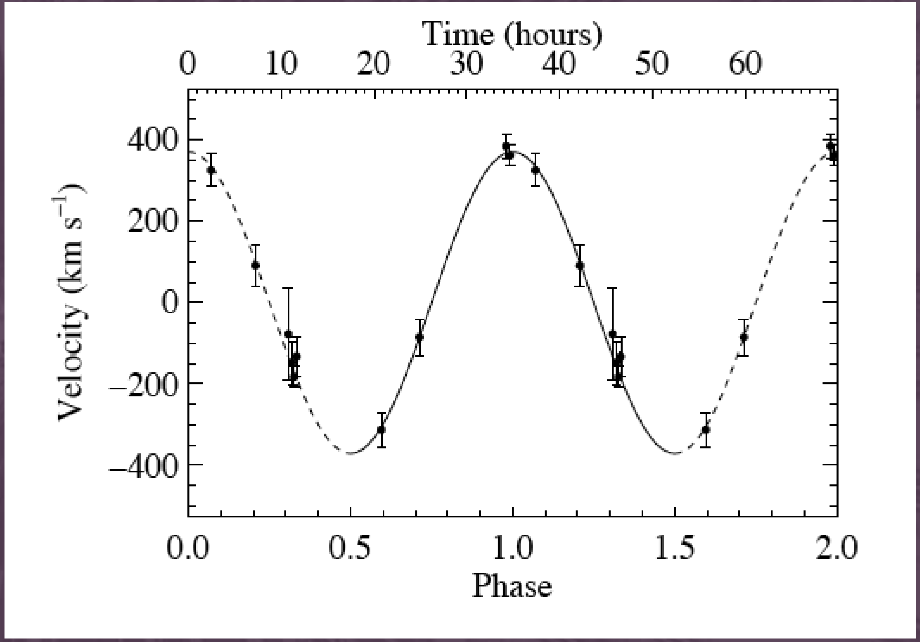
See Thorne Fig 8.3





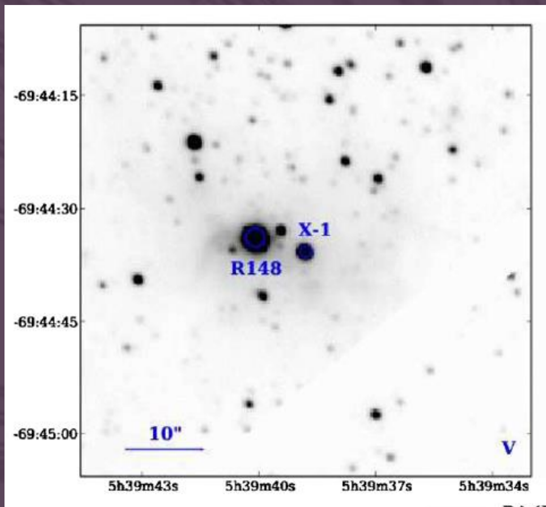
Silverman & Filippenko 2008, confirmation of 33 ± 3 $M_\odot$ black hole in a nearby galaxy (IC 10),

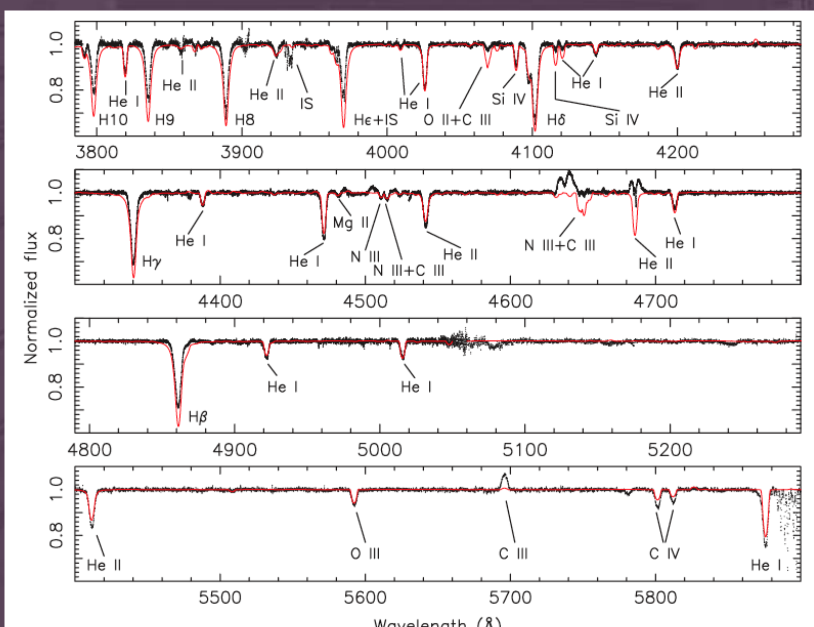


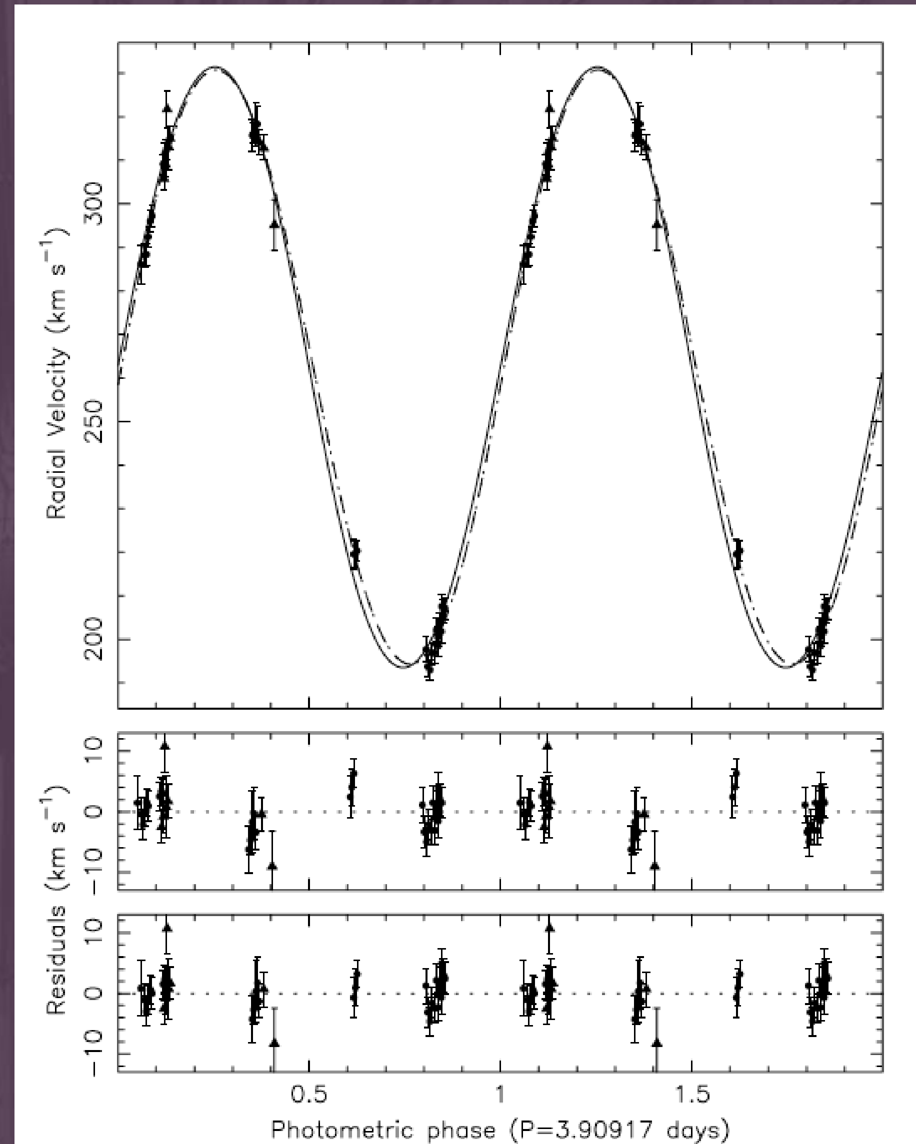


Helium emission line

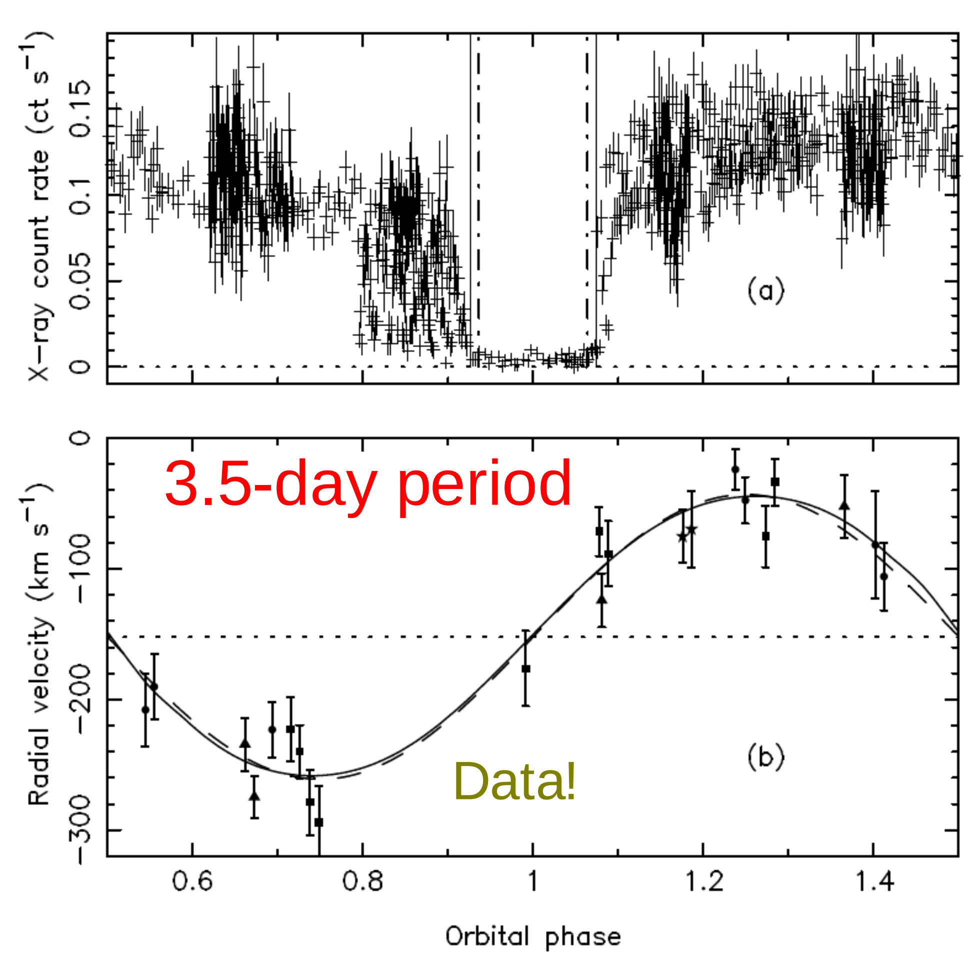
Orosz et al. 2009, LMC X-1, $M_{BH} = 10.9 \pm 1.4\ M_{\odot}$

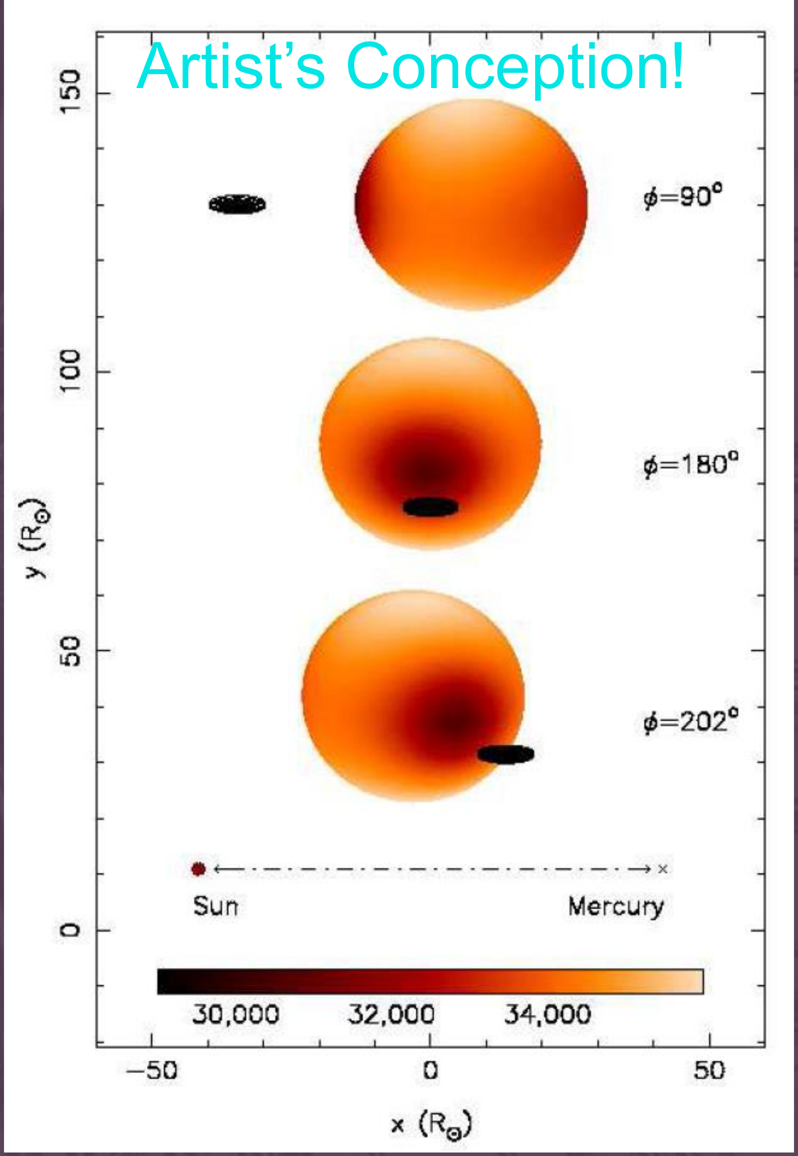


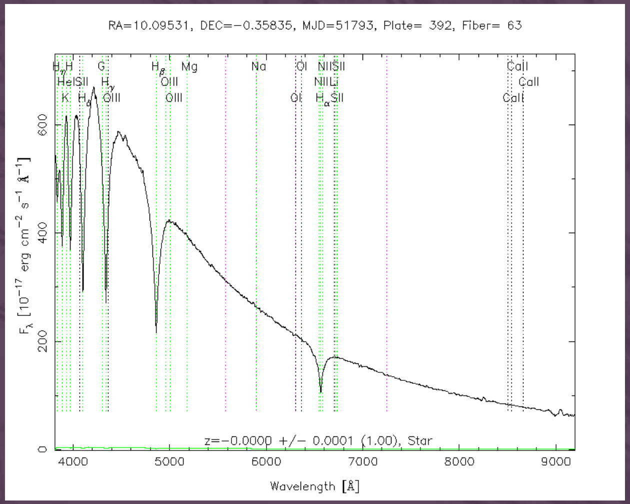


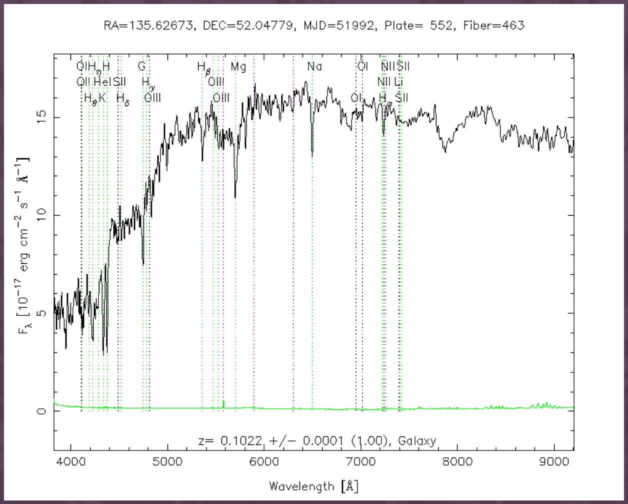


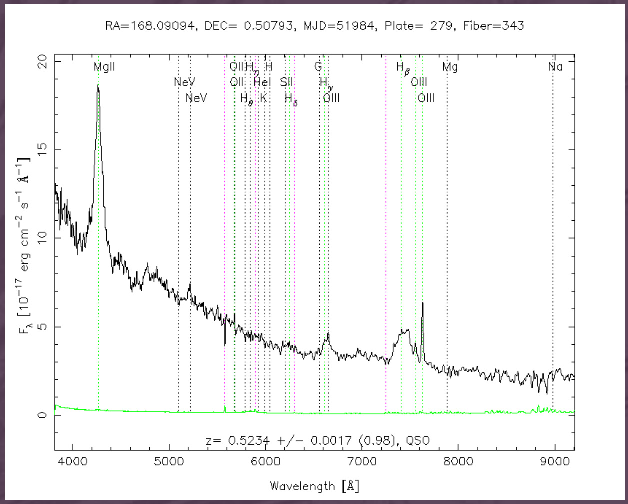
Orosz et al. 2007, a 15.7$M_\odot$ BH in nearby galaxy M33, eclipsed by its 70 $M_\odot$ companion







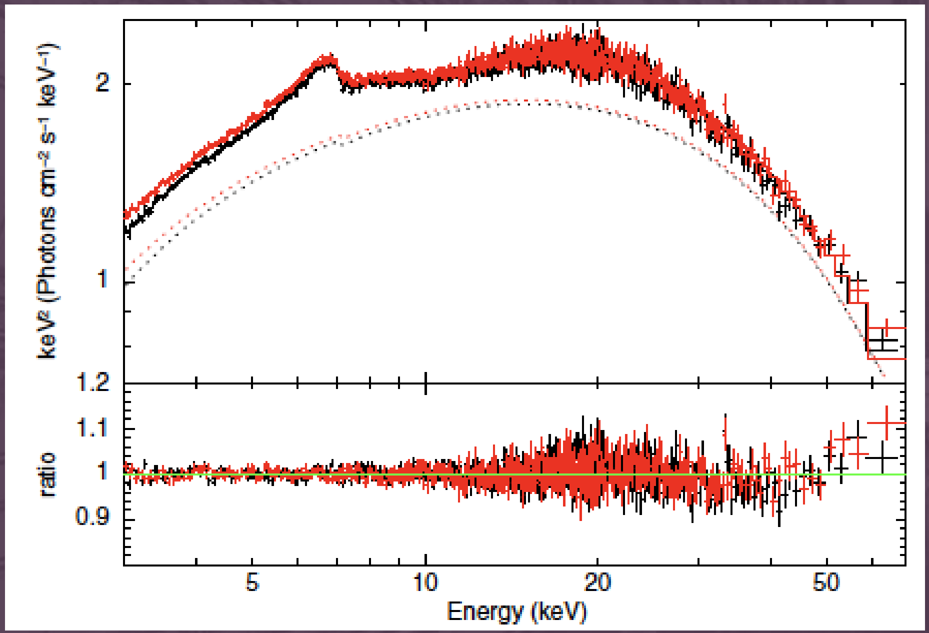


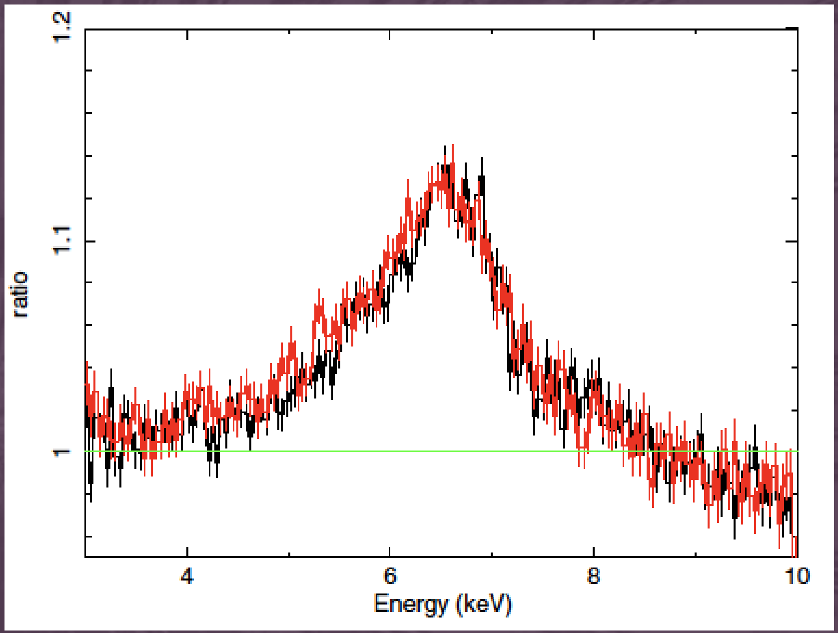


White dwarf

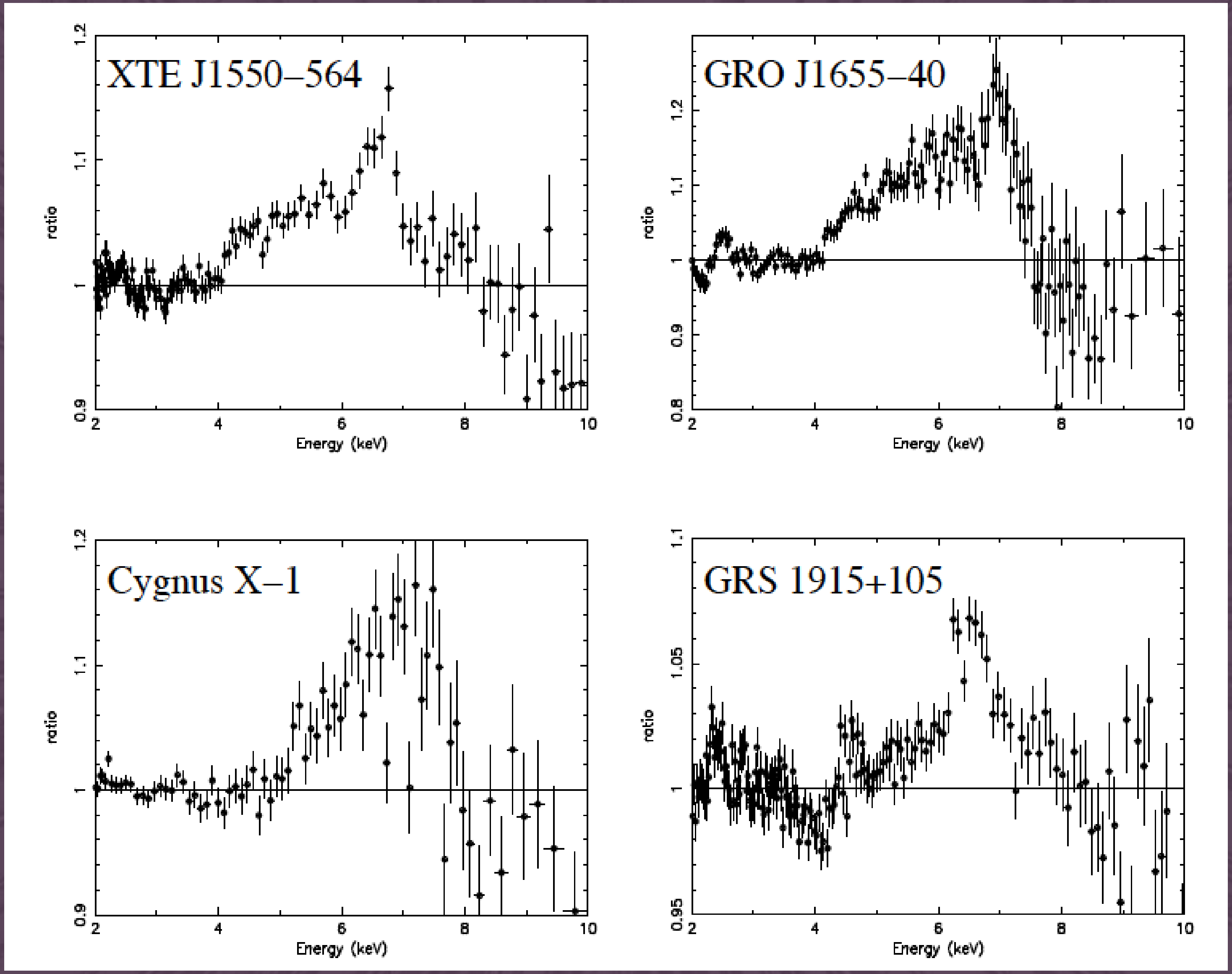
Galaxy

Quasar



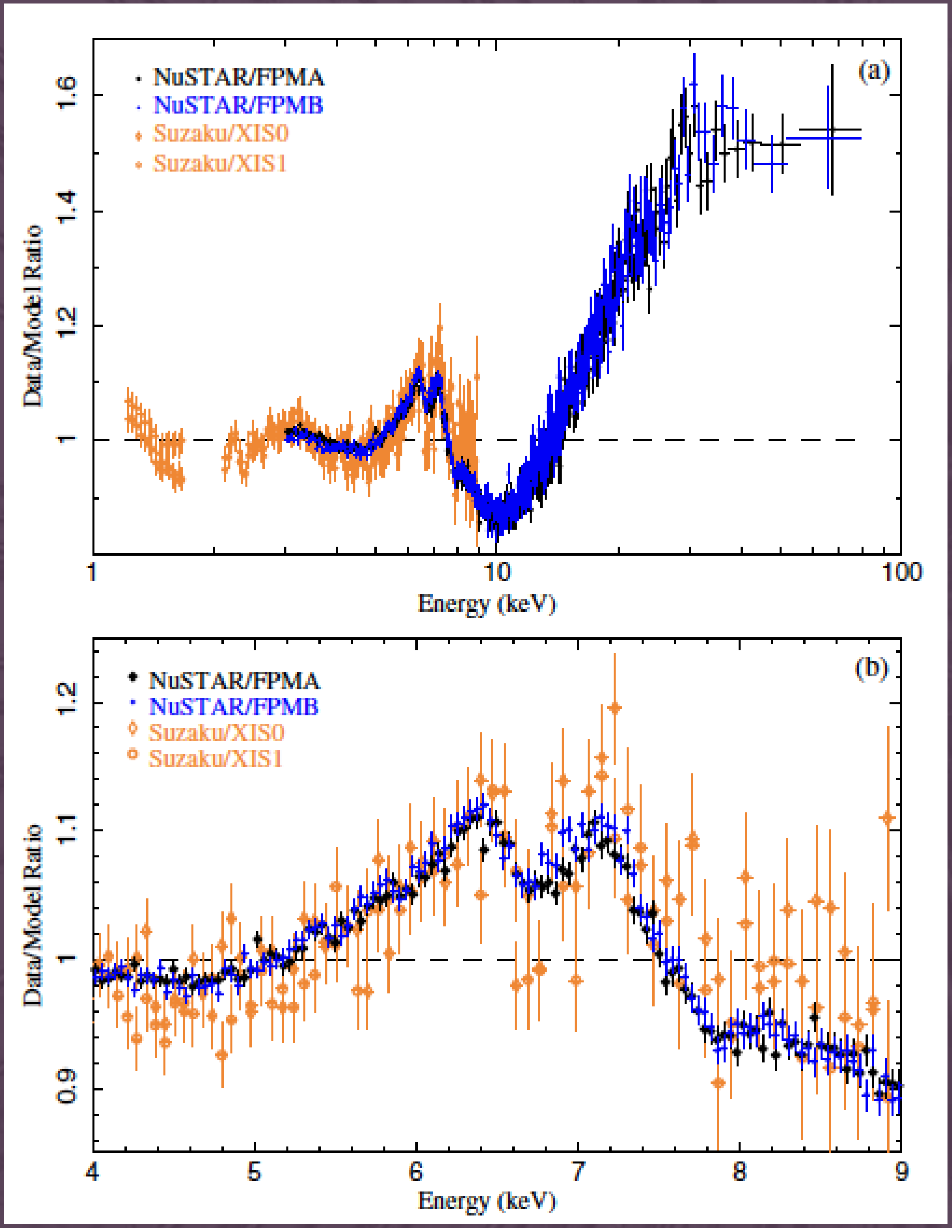


X-ray spectra of stellar BH candidate GRS1739-278
Miller et al. 2015



X-ray iron lines of four stellar mass black holes

From Miller 2007, ARAA



X-ray spectrum of stellar mass black hole Cygnus X-1.

Broad iron line shows evidence of high BH spin and absorption at 6.7 keV.

Tomsick et al. 2014