A5682: Introduction to Cosmology

**Metrics and Metric Notation: Examples**

A metric is a matrix (more precisely a tensor) of functions that tells how to go from differential coordinate separations to differential distances (which may be spatial distances or spacetime intervals).

First let's consider two-dimensional space metrics. The spatial separation of nearby points can be written

$$(dl)^2 = \sum_{i,j=1,2} g_{ij} dx_i dx_j \ .$$

Start with Cartesian coordinates on a flat plane, and identify $x_1 = x$, $x_2 = y$. Standard differential geometry notation would use superscripts here rather than subscripts, but I've stuck with subscripts to avoid confusing them with powers.

In this case the metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

yielding

$$(dl)^2 = 1 \cdot dx_1 dx_1 + 0 \cdot dx_1 dx_2 + 0 \cdot dx_2 dx_1 + 1 \cdot dx_2 dx_2 = (dx)^2 + (dy)^2 \ .$$

For polar coordinates on a flat plane we identify $x_1 = r$, $x_2 = \theta$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

yielding

$$(dl)^2 = 1 \cdot dx_1 dx_1 + r^2 \cdot dx_2 dx_2 = (dr)^2 + r^2 (d\theta)^2 \ .$$

These examples illustrate two important points. First, even if the underlying space is the same, the metric depends on the coordinate system used to describe it. Second, the metric is in general a function of position (e.g., depending on $r$ in the second case). A constant metric is a special case.

Both of these examples have diagonal metrics. However, if the axes were not orthogonal, then the metric would have off-diagonal terms. The metric is always a symmetric matrix because $dx_i dx_j = dx_j dx_i$.

For polar coordinates on a 2-d spherical surface of curvature radius $R$, the metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin^2\left(\frac{r}{R}\right) \end{pmatrix} \ .$$

Now consider flat four-dimensional spacetime, with Cartesian spatial coordinates:

$$x_0 = t, \qquad x_1 = x, \qquad x_2 = y, \qquad x_3 = z \ .$$

By convention, the coordinate subscripts for 4-d spacetime are denoted with Greek rather than Latin subscripts.

We are now interested in the flat-spacetime interval separation

$$(ds)^2 = \sum_{\mu,\nu=0,3} g_{\mu\nu} dx_\mu dx_\nu \ .$$

In these coordinates, the metric is

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

yielding

$$\begin{aligned} (ds)^2 &= -c^2 \cdot dx_0 dx_0 + 1 \cdot dx_1 dx_1 + 1 \cdot dx_2 dx_2 + 1 \cdot dx_3 dx_3 \\ &= -c^2 (dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \ . \end{aligned}$$

This is called the Minkowski metric, which can also be written in spherical coordinates.

Relativists frequently work in units where $c = 1$ by definition.

In these units the metric for flat spacetime in Cartesian coordinates is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$