Monday, February 6
Friedmann Equation

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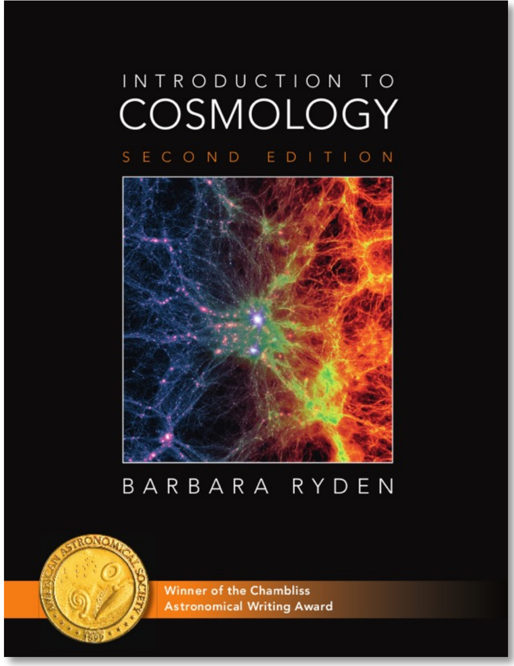
Christine de Pizan, *The Book of the Queen*, 1412 (Master of the *Cité des Dames*, illustrator)



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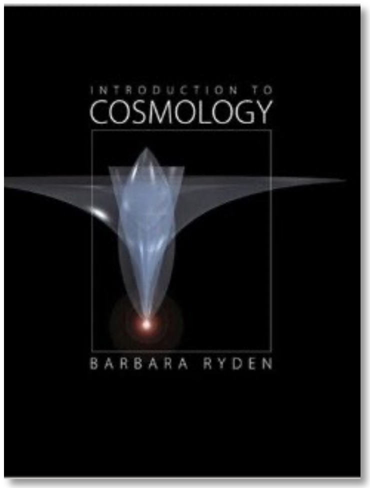
(Andrew Ward, cover designer)

Introduction to Cosmology

1st edition



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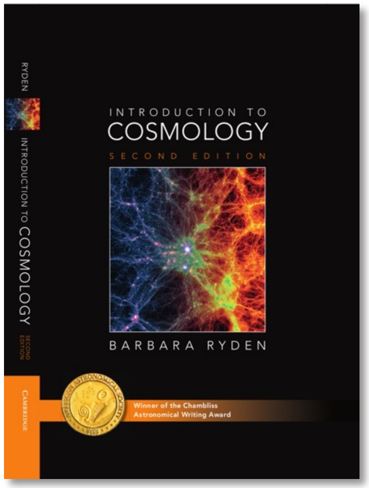

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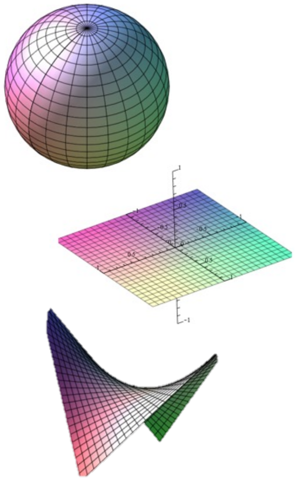
Assume: Space is homogeneous & isotropic.
Expansion of space is homogeneous & isotropic.

The result is the Robertson-Walker metric:

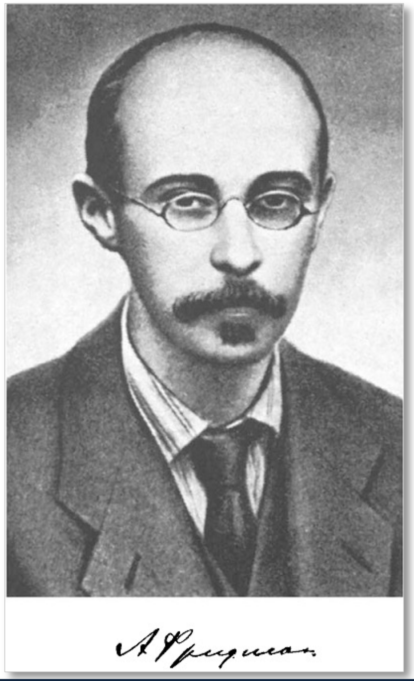
$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2[\mathrm{d}r^2 + S_\kappa(r)^2\mathrm{d}\Omega^2]$$

$$a(t) = \text{scale factor}$$

$$S_\kappa(r) = \begin{cases} R_0\sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0\sinh(r/R_0) & \kappa = -1 \end{cases}$$

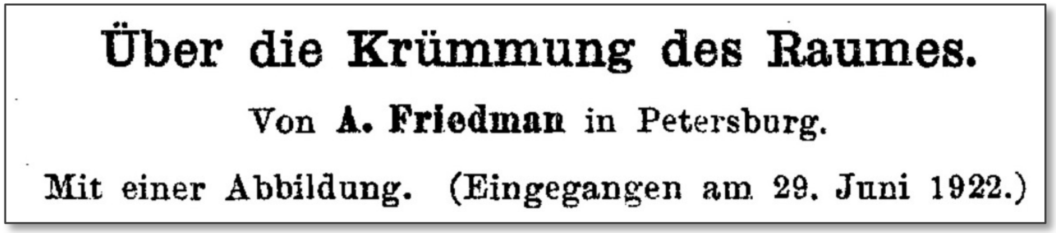


How do the geometric properties of the universe [scale factor $a(t)$, curvature κ, radius of curvature R_0] depend on its mass-energy density $\varepsilon(t)$?



Александр Фридман

Answer found in 1922 by Alexander Friedmann, starting from Einstein's field equation.

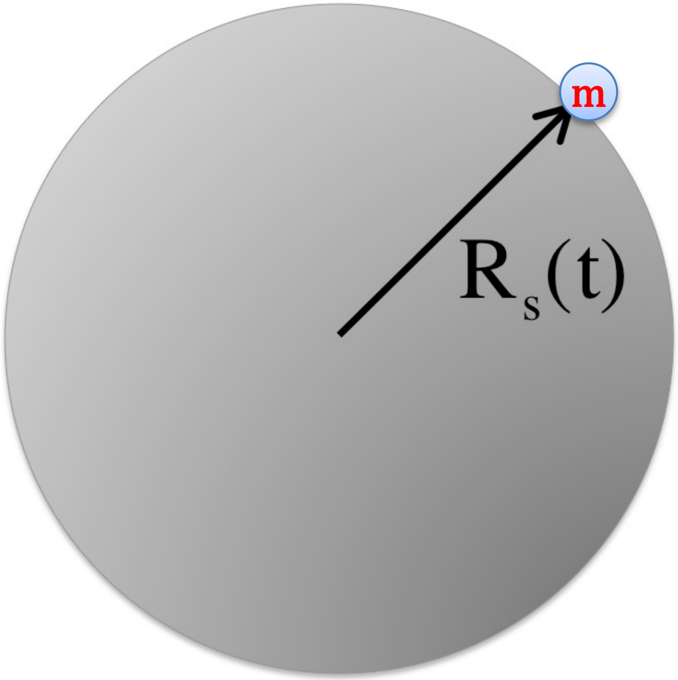


Über die Krümmung des Raumes.

Von **A. Friedman** in Petersburg.

Mit einer Abbildung. (Eingegangen am 29. Juni 1922.)

Newtonian equivalent of Friedmann equation



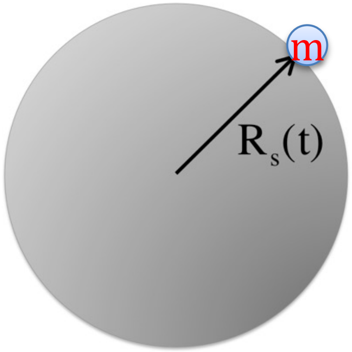
Uniform density sphere

mass M_s

radius $R_s(t)$

test mass m is placed at its surface.

Newtonian gravity is the only force at work.



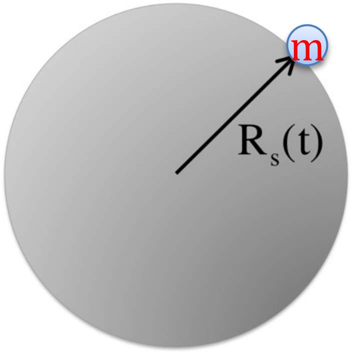
Force on test mass

$$F = -\frac{GM_s m}{R_s^2}$$

Acceleration of test mass

$$\frac{d^2R_s}{dt^2} = \frac{F}{m} = -\frac{GM_s}{R_s(t)^2}$$

If the whole sphere contracts (or expands), its mass M_s remains constant as R_s decreases (or increases).

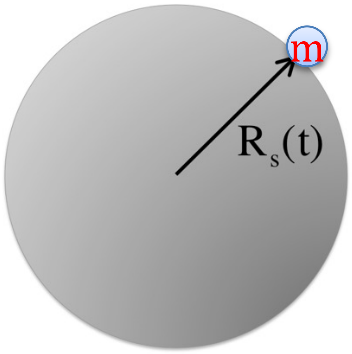


$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}$$

$$\frac{dR_s}{dt}\cdot\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}\cdot\frac{dR_s}{dt}$$

let $v_s \equiv \dfrac{dR_s}{dt}$ ☛ $v_s\dfrac{dv_s}{dt} = -\dfrac{GM_s}{R_s^2}\dfrac{dR_s}{dt}$

$$\int v_s dv_s = -GM_s\int\frac{dR_s}{R_s^2}$$



$$\int v_s dv_s = -GM_s \int \frac{dR_s}{R_s^2}$$

$$\frac{1}{2}v_s^2 = \frac{GM_s}{R_s} + U$$

constant of integration

kinetic energy *per unit mass* of test particle

− potential energy *per unit mass* of test particle

If gravity is the only force, **kinetic energy + gravitational potential energy** is conserved.

$$\frac{1}{2}v_s^2 = \frac{GM_s}{R_s} + U$$

Mass is conserved.

$$M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3$$

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{4\pi}{3}G\rho(t)R_s(t)^2 + U$$

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{4\pi}{3}G\rho(t)R_s(t)^2 + U$$

Let’s get cosmological, with homogeneous and isotropic expansion:

$$R_s(t) = a(t)r_s$$

scale factor

comoving radius of sphere

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}G\rho(t)r_s^2 a(t)^2 + U$$

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}G\rho(t)r_s^2a(t)^2 + U$$

Scale factor $a(t)$ is linked to mass density $\rho(t)$ by conservation of energy and mass.

Tidy up, and divide by $r_s^2a^2/2$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

This is the Friedmann equation in its **Newtonian** form.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

Time derivative of $a(t)$ enters only as $\dot{a}^2$

Expanding sphere $(\dot{a} > 0)$ is perfect time reversal of contracting sphere $(\dot{a} < 0)$.

No friction, no air resistance, no increase in entropy.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

never negative

never negative

can be negative, zero, or positive (depends on sign of U)

The fate of an *expanding* sphere depends on U.

$$U = \frac{1}{2}v_s^2 - \frac{GM_s}{R_s}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

$$\boxed{U < 0}$$

Sphere is initially expanding (both sides of equation are initially positive).

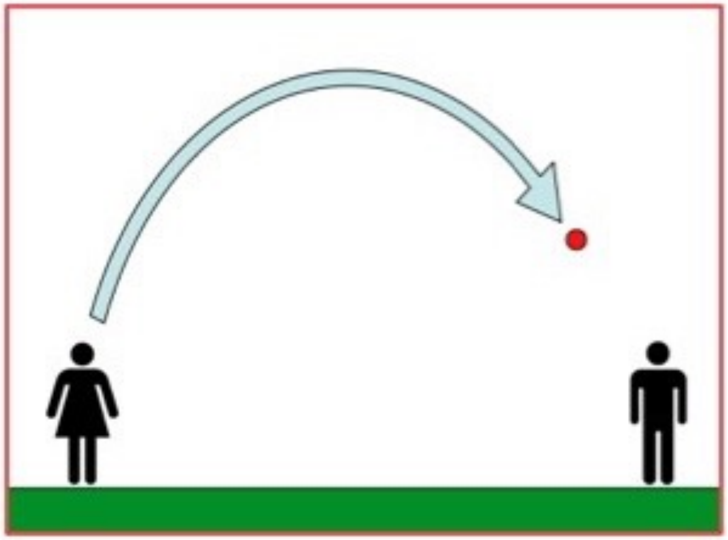
Right hand side equals zero when

$$a = a_{max} = -\frac{GM_s}{Ur_s}$$

Sphere stops expanding.

Sphere starts to contract.

$U < 0$: Analogous to tossing a ball upward with speed less than the escape speed.



What goes up must come down.

...UNLESS it is moving faster than the escape speed,

$$v_{esc} = \sqrt{\frac{2GM_s}{R_s}}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

$$\boxed{U > 0}$$

Right hand side of equation is *always* positive.

$\dot{a}^2$ is *always* positive.

Initially expanding sphere *keeps expanding*.

(Analogy: toss a ball upward
faster than the escape speed.)

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \cancel{\frac{2U}{r_s^2}\frac{1}{a(t)^2}}}$$

$U = 0$: Boundary case where expansion speed *exactly equals* the escape speed.

$\dot{a} \to 0$ as $t \to \infty$ and $\rho \to 0$.

We ❤️ Newton, but...
a *full* derivation of the Friedmann equation requires general relativity (GR).

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

Einstein says: Mass and energy are equivalent.
Replace mass density $\rho(t)$ [kg/m^3]
with mass-energy density $\varepsilon(t)$ [J/m^3]

Step one:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

Einstein says: Gravity is not a force.
Gravity is motion along geodesics
in a curved 4-d space-time.

Gravitational potential energy
is not a useful concept in GR.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) \boxed{+ \frac{2U}{r_s^2}\frac{1}{a(t)^2}}$$

This term is not
GR-friendly.

"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Step two: going from Newtonian physics to GR:

$$\frac{2U}{r_s^2} \rightarrow -\frac{\kappa c^2}{R_0^2}$$

$U < 0$ (recollapsing) $\rightarrow$ $\kappa = +1$ (positively curved)

$U > 0$ (eternal expansion) $\rightarrow$ $\kappa = -1$ (negatively curved)

$U = 0$ (exactly at escape speed) $\rightarrow$ $\kappa = 0$ (exactly flat)

The One True Friedmann Equation (Einstein-approved):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

A little more simply:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter.

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

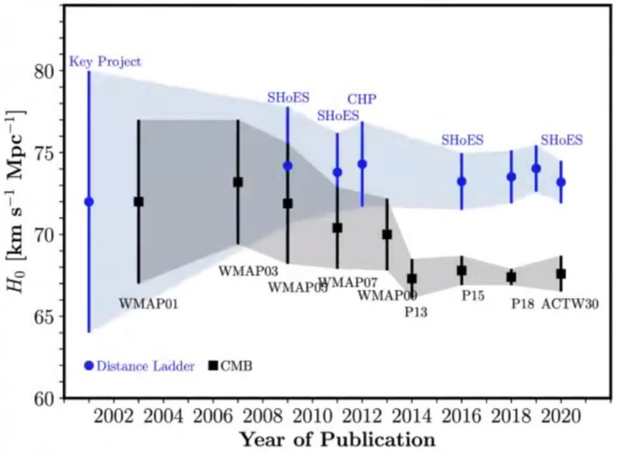
At the present day $(t = t_0)$,

$$a(t_0) = 1$$

$$\varepsilon(t_0) = \varepsilon_0$$

$H(t_0) = H_0 =$ the **Hubble constant**

The value of the Hubble constant is a matter of some debate.



$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

expansion rate

mass-energy

curvature

$$H_0 \approx 70 \text{ km s}^{-1}\text{Mpc}^{-1}$$

If we measure κ and R_0, we can find ε_0.

In theory, if we measure ε_0, we can find κ and R_0. However, doing an inventory of mass-energy is ***really freaking difficult!***

In a flat universe $(\kappa = 0)$, the Friedmann equation is delightfully simple:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t)$$

For a given value of the Hubble parameter, there is a critical density ε_c at which the universe is flat.

$$\varepsilon_c(t) \equiv \frac{3c^2}{8\pi G}H(t)^2$$

What's the critical density *right now*?

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2$$

$$\varepsilon_{c,0} = 8.27\times10^{-10}\,\mathrm{J\,m^{-3}}\,h_{70}^2$$

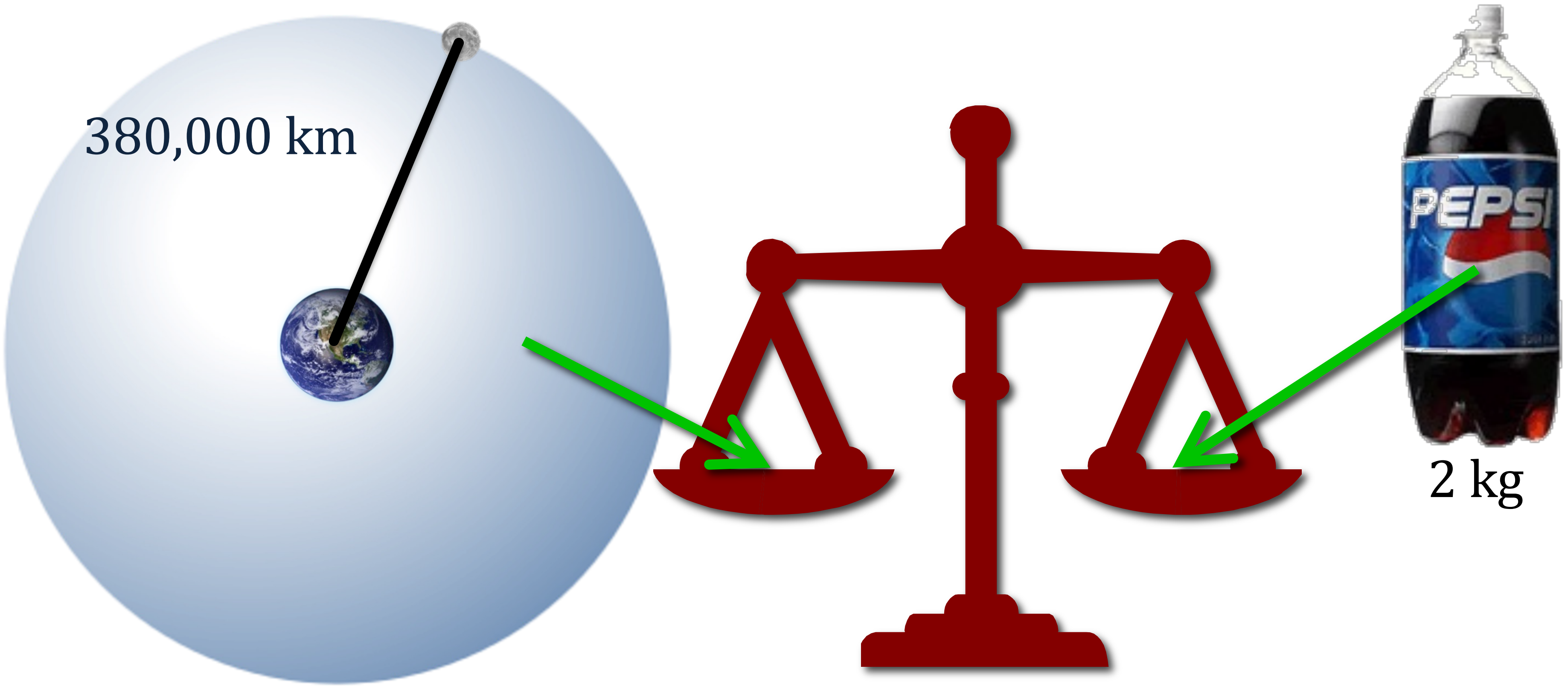
$$h_{70} \equiv H_0/70\,\mathrm{km\,s^{-1}Mpc^{-1}}$$

$$\varepsilon_{c,0} = 5160\,\mathrm{MeV\,m^{-3}}\,h_{70}^2$$

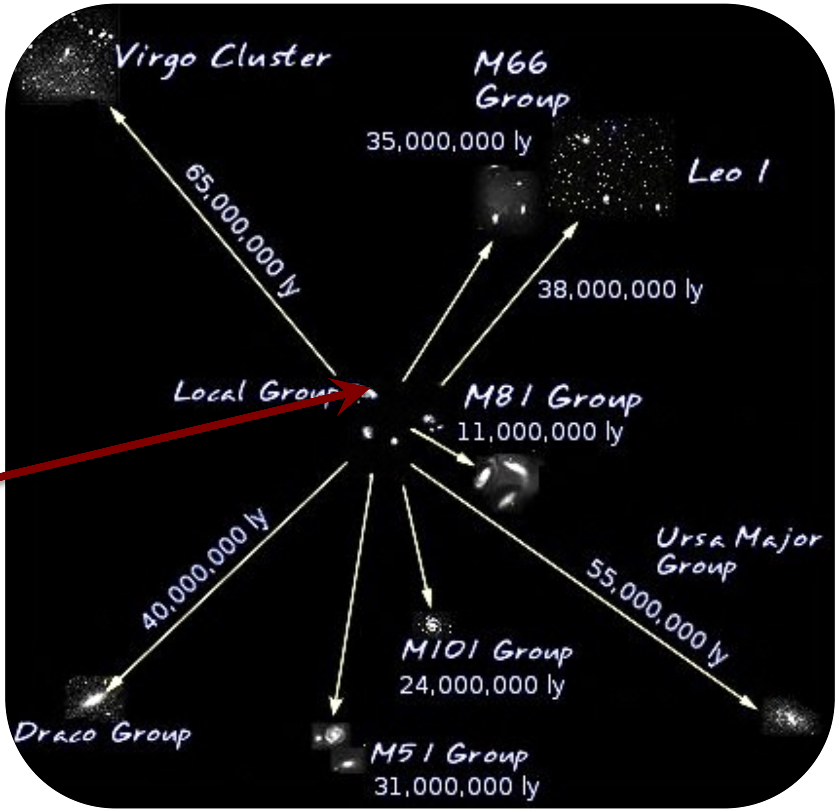
$$\rho_{c,0} = \varepsilon_{c,0}/c^2 = 9.20\times10^{-27}\,\mathrm{kg\,m^{-3}}\,h_{70}^2$$

$$\rho_{c,0} = 1.36\times10^{11}\,M_\odot\,\mathrm{Mpc^{-3}}\,h_{70}^2$$

The critical density is absurdly low by terrestrial standards.



The critical density seems absurdly low. However, most of the universe consists of absurdly empty intergalactic voids.



You are here

Fun with dimensionless numbers!

Cosmologists express the density of the universe in terms of the density parameter **Ω**

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

actual density

critical density

$\Omega = 1$ flat (Euclidean)

$\Omega < 1$ negative curvature

$\Omega > 1$ positive curvature