3 Hydrostatic Equilibrium

Reading: Shu, ch. 5, ch. 8

3.1 Timescales and Quasi-Hydrostatic Equilibrium

Consider a gas obeying the Euler equations:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\vec{\nabla}\cdot\vec{u}, \;\; \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \vec{g} - \frac{1}{\rho}\vec{\nabla}P, \;\; \frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u} + \frac{\Gamma-\Lambda}{\rho}.$$

Suppose that there is a substantial mismatch between the gravitational acceleration and the pressure gradient

$$|\vec{g} - \frac{1}{\rho}\vec{\nabla}P| \sim |\vec{g}|.$$

How long does it take for this mismatch to produce an order unity change in the density?
In time Δt

$$\frac{\Delta\rho}{\rho} \sim -(\vec{\nabla}\cdot\vec{u})\Delta t \sim (-\vec{\nabla}\cdot(\vec{g}\Delta t))\Delta t.$$

If the gas is self-gravitating, then

$$\vec{\nabla}\cdot\vec{g} = -\nabla^2\phi = -4\pi G\rho, \text{ so}$$

$$\frac{\Delta\rho}{\rho} \sim 4\pi G\rho(\Delta t)^2.$$

The density therefore changes on a dynamical timescale

$$\Delta t \sim t_d \sim (G\rho)^{-1/2}.$$

Binney & Tremaine (eq. 2-30) define

$$t_d \equiv \left(\frac{3\pi}{16G\rho}\right)^{1/2} \approx \frac{3}{4}(G\rho)^{-1/2}.$$

The relevant ρ is that of the gravitationally dominant component.
Since pressure is an increasing function of density, these changes generally go in the direction of restoring balance between gravity and pressure gradients.

If a gas dynamical system is many dynamical times old, we generally expect it to be in hydrostatic equilibrium, with

$$\vec{g} = \frac{1}{\rho}\vec{\nabla}P.$$

Suppose that the gas is cooling, so that the pressure drops over time. On what timescale does the system evolve?
Slow cooling: cooling time $t_c^{-1} \sim \frac{1}{\epsilon}\frac{\mathrm{D}\epsilon}{\mathrm{D}t}$.

Rapid cooling: dynamical time.

The timescale for cooling, or for other processes that change the pressure, is often much longer than the dynamical timescale. If a system has balance between gravity and pressure gradients but evolves on a timescale $\gg t_d$, we say it is in quasi-hydrostatic equilibrium. For such a system, we ignore time derivatives in the momentum equation but not in the other equations.

3.2 Isothermal plane-parallel atmospheres

For a plane-parallel atmosphere in a uniform gravitational field, $\vec{g} = g\hat{\mathbf{z}}$, hydrostatic equilibrium implies

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -g.$$

With $P = (\rho/m)kT$ we obtain

$$\frac{\mathrm{d}\ln P}{\mathrm{d}z} = \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{-gm}{kT},$$

with solution

$$\begin{aligned} P &= P_0 e^{-z/z_h}, \qquad & z_h &\equiv \frac{kT}{gm} \\ \rho &= \rho_0 e^{-z/z_h}, \qquad & \rho_0 &\equiv P_0\left(\frac{m}{kT}\right). \end{aligned}$$

For the earth, $g = 980\,\mathrm{cm\,s^{-2}}$, $m \approx 4.8\times10^{-23}\,\mathrm{g}$, $T \approx 290\,\mathrm{K}$, $P_0 \approx 10^6\,\mathrm{dyne\,cm^{-2}}$, $\rho_0 \approx 1.2\times10^{-3}\,\mathrm{g\,cm^{-3}}$, $z_h \approx 8.5\,\mathrm{km}$.
Implication: it's hard to breathe on top of Mount Everest.

We can get an order-of-magnitude estimate of the scale height from a very simple argument based on the analogous collisionless case.
The one-dimensional velocity dispersion of particles is $\sigma^2 = kT/m$, since the mean kinetic energy per translational degree of freedom, $\frac{1}{2}m\sigma^2$, should equal $\frac{1}{2}kT$.
A particle up from $z = 0$ with velocity σ comes to rest after time $t \sim \sigma/g$, having traveled a distance $h \sim \sigma t \sim \sigma^2/g = kT/(gm)$.

If we had assumed an adiabatic atmosphere instead of an isothermal atmosphere, with $P = P_0(\rho/\rho_0)^\gamma$, we would have obtained a very different result:

$$P(z) = P_0\left(1 - \frac{z}{z_a}\right)^{\frac{\gamma}{\gamma-1}}, \quad \rho(z) = \rho_0\left(1 - \frac{z}{z_a}\right)^{\frac{1}{\gamma-1}}, \quad z_a = 30\,\mathrm{km}.$$

What do we make of the negative pressure that comes from this solution at $z > z_a$?
Underlying equations break down as $\rho \longrightarrow 0$, corresponding to $\lambda \longrightarrow \infty$.

3.3 Scale-height of a thin disk

Consider a disk of gas orbiting at radius r around a central mass providing gravitational acceleration $g = v_\phi^2/r$.
For gas at height $z \ll r$ above the midplane, the z-component of gravitational acceleration is $g_z = -gz/r$.
Thus

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{g}{r}z,$$

and

$$\frac{\mathrm{d}\ln P}{\mathrm{d}z} = \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{g}{r\sigma^2}z,$$

with $\sigma^2 = kT/m$.
The solution is $P = P_0 e^{-z^2/z_h^2}$ with

$$z_h = \sigma\left(\frac{2r}{g}\right)^{1/2}.$$

Substituting $g = v_\phi^2/r$ yields

$$\frac{z_h}{r} = \sqrt{2}\frac{\sigma}{v_\phi}.$$

This is again what one would expect in order-of-magnitude from the collisionless case, where a particle goes a vertical distance z_h at vertical speed σ in about the same time that it goes an azimuthal distance r at azimuthal speed v_ϕ.

3.4 Isothermal sphere in hydrostatic equilibrium

Good discussions in Binney & Tremaine §4.4(b); Shapiro, Iliev, & Raga 1999 (MNRAS, 307, 203)

Consider a self-gravitating spherical distribution of gas with temperature T.
Hydrostatic equilibrium with $P = nkT = (\rho/m)kT$ requires

$$\vec{\nabla}P = \rho\vec{g} \quad \Longrightarrow \quad \frac{\mathrm{d}P}{\mathrm{d}r} = \frac{kT}{m}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho\frac{GM(r)}{r^2},$$

and

$$M(r) = \int_0^r 4\pi r^2 \rho dr \quad \Longrightarrow \quad \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho.$$

Multiply by $\frac{r^2 m}{\rho kT}$ to get

$$r^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{Gm}{kT}M(r)$$

and differentiate to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) = -4\pi\frac{Gm}{kT}r^2\rho, \tag{25}$$

which can be integrated to obtain $\rho(r)$.

Singular isothermal sphere (SIS) solution:

Assume $\rho = Cr^{-b}$, implying

$$\ln \rho = \ln C - b \ln r \quad \Longrightarrow \quad \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} = -\frac{b}{r}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(-br) = -b = -4\pi \frac{Gm}{kT} C r^{2-b}$$

$$b = 2, \quad C = \frac{kT}{2\pi Gm}$$

$$\rho(r) = \frac{kT}{2\pi Gm} r^{-2} = \frac{\sigma^2}{2\pi G r^2}$$

where, as usual, $\sigma = \left(\frac{kT}{m}\right)^{1/2}$.
The mass interior to r is

$$M(r) = \frac{2\sigma^2}{G} r$$

and the circular velocity is

$$v_c^2 = \frac{GM(r)}{r} = 2\sigma^2.$$

An isothermal (constant σ^2) stellar dynamical (collisionless) sphere obeys the same equation and has the same structure.

The solution to the differential equation (25) depends on the boundary condition as well as the equation itself.
The SIS solution has a cusp at $r = 0$.
What about the solution for finite central density ρ_0?

First, we would like to identify a characteristic lengthscale so that we can obtain a general solution in terms of dimensionless variables.
The dimensional quantities in equation (25) are G, ρ, and the combination $\sigma^2 = kT/m$.
The combination of these quantities that yields a lengthscale is $\sigma/(G\rho_0)^{1/2}$.
Physically, this represents the typical distance a particle travels in a central dynamical time.
Define the King radius (a.k.a. core radius)

$$r_0 \equiv \left(\frac{9\sigma^2}{4\pi G \rho_0}\right)^{1/2} \tag{26}$$

and dimensionless variables

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{r} = \frac{r}{r_0}.$$

The hydrostatic equilibrium equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}} \left(\tilde{r}^2 \frac{\mathrm{d} \ln \tilde{\rho}}{\mathrm{d}\tilde{r}}\right) = -9 \tilde{r}^2 \tilde{\rho}. \tag{27}$$

The (numerical) solution is obtained by integrating outwards from $\tilde{r} = 0$ with the central boundary conditions

$$\tilde{\rho}(0) = 1, \qquad \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\tilde{r}} = 0$$

(the second condition is required for hydrostatic equilibrium, since the interior mass vanishes as $\tilde{r} \longrightarrow 0$).

The solution bends from a constant $\tilde{\rho}$ at $r \lesssim 1$ to the singular isothermal sphere solution $\tilde{\rho} = \frac{2}{9}\tilde{r}^{-2}$ at $\tilde{r} \gtrsim 3$. For $\tilde{r} \lesssim 2$, the solution is well described by the simple approximation

$$\tilde{\rho} = (1 + \tilde{r}^2)^{-3/2},$$

which is accurate to better than 5%, but this approximation fails at $r \gtrsim 3$ (it has the wrong asymptotic slope).

The *projected* surface density of an isothermal sphere drops to $0.5013 \approx 0.5$ of its central value at $\tilde{r} = 1$.

Note that by defining characteristic dimensionless variables we have reduced the *family* of solutions with different central densities and temperatures to a single solution for the appropriately scaled variables.

What is the total mass of an isothermal sphere of central density ρ_0 and temperature T? Infinite. In order to get a system of finite total mass, we must truncate it at radius r_t by confining it with an external pressure.

3.5 Polytropes

A spherical, self-gravitating object in hydrostatic equilibrium with a polytropic equation of state $P = K\rho^\gamma$ is called a polytrope. The structure of a polytrope is determined by the adiabatic index γ or by the so-called polytropic index $n \equiv (\gamma - 1)^{-1}$.

Consider a polytrope with equation of state

$$P = K\rho^\gamma = \left(\frac{P_c}{\rho_c^\gamma}\right)\rho^\gamma = P_c\left(\frac{\rho}{\rho_c}\right)^{1+1/n},$$

where P_c and ρ_c are the central pressure and temperature.
The quantity

$$\alpha \equiv \left[\frac{\gamma}{\gamma - 1}\frac{P_c}{\rho_c}\frac{1}{4\pi G\rho_c}\right]^{1/2}$$

has units of length. We will soon learn that the sound speed is $a = (\gamma P/\rho)^{1/2}$, so α is proportional to the speed of sound times the dynamical time in the center of the polytrope.

In terms of the dimensionless variables θ and ξ defined by

$$\rho = \rho_c\theta^n, \qquad P = P_c\theta^{n+1}, \qquad r = \alpha\xi,$$

the equation of hydrostatic equilibrium can be written

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n,$$

called the Lane-Emden equation for a polytropic star.
The structure of a non-singular polytrope can be found by integrating this equation with the central boundary conditions

$$\theta = 1, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = 0 \qquad \text{at } \xi = 0.$$

The isothermal sphere corresponds to $n = \infty$. For $n \geq 5$ the total mass is infinite. For $n < 5$, the density drops to zero at a finite radius.
The cases most interesting for stars are $n = 3/2$ ($\gamma = 5/3$) and $n = 3$ ($\gamma = 4/3$), but these do not have analytic solutions.
The only analytic solutions are for $n = 1$ and $n = 5$. For $n = 1$ the solution is

$$\theta = \frac{\sin\xi}{\xi}.$$

Generically, polytropes with $n < 5$ have a roughly constant density core whose radius is $\sim \alpha$ followed by a falling density profile that eventually drops to zero.
For higher n (lower γ), the polytrope is more centrally concentrated (smaller core) because the "squishier" (more compressible) equation of state means that a higher central density is required to support the weight of the layers above.

3.6 Instability considerations

Two of the instabilities that can affect a fluid in hydrostatic equilibrium are Rayleigh-Taylor instability and convective instability.

Rayleigh-Taylor instability occurs when a dense fluid sits on top of a light fluid.
It is energetically favorable for dense fluid elements to sink and exchange places with light fluid elements.
Although a true hydrostatic equilibrium situation with Rayleigh-Taylor instability is unlikely to arise in astrophysical situations, the instability can be important in, e.g., outflows or supernova explosions when a light medium tries to accelerate a dense medium.

Convective instability arises if a fluid element that moves upward (against gravity) and expands adiabatically (because it doesn't have time to exchange heat conductively with its surroundings) becomes less dense than its new surroundings, so that it continues to rise.
The *Schwarzschild criterion* that determines whether a fluid is convectively unstable is

$$ds > 0 \quad \text{in the direction of gravity.}$$

High-entropy material is buoyant and tends to move past low-entropy material.