Radiative Gas Dynamics

Problem Set 2: Isothermal Spheres

Part I, due Thursday, Jan. 25; Part II, due Tuesday, Jan. 30

Part I: Singular Isothermal Spheres and the Virial Theorem

(a) Show that a spherical, self-gravitating object in hydrostatic equilibrium satisfies the 2nd-order differential equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{1}$$

(b) Show that for a gas of constant temperature T and particle mass m, equation (1) can be written

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) = -4\pi\frac{Gm}{kT}r^2\rho. \tag{2}$$

Show that the density profile

$$\rho(r) = \frac{kT}{2\pi Gm}r^{-2} \tag{3}$$

is a solution to this equation. Equation (3) is the density profile of a singular isothermal sphere.

(c) Consider a singular isothermal sphere of temperature T and particle mass m (i.e., $m = m_p$ for hydrogen). Assume that the sphere has a finite total mass M because it is truncated at radius R by being confined in a surrounding external medium of pressure $P_{\rm ext}$.

What is R in terms of M, m, and T?

What is $P_{\rm ext}$ in terms of T, m, and R?

(d) Use the hydrostatic equilibrium equation to show that any hydrostatic spherical system of radius R in an external medium of pressure $P_{\rm ext}$ satisfies

$$2U_{\rm kin} + W + S_p = 0, \tag{4}$$

where

$$U_{\rm kin} = \int_0^M \frac{3}{2}\frac{kT}{m}dM = \int_0^M \frac{3}{2}\frac{P}{\rho}dM \tag{5}$$

is the kinetic energy of thermal motion,

$$W = -\int_0^M \frac{GM(r)dM}{r} \tag{6}$$

is the gravitational potential energy, and

$$S_p = -4\pi R^3 P_{\rm ext}.$$

Note that equation (4) becomes the more familiar and memorable form of the virial theorem, $2U + W = 0$, if and only if the gas is monatomic (so that $U_{\rm kin} = U$ is the total thermal energy) and the external pressure is zero (as it would be for a star).

(e) Evaluate W, $U_{\rm kin}$, and S_p for the truncated singular isothermal sphere of part (a) and verify explicitly that it satisfies the virial theorem (4).

Part II: Structure of Non-Singular Isothermal Spheres

As discussed in class, the differential equation that describes a non-singular isothermal sphere is

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\frac{\tilde{r}^2}{\tilde{\rho}}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\tilde{r}}\right) = -9\tilde{r}^2\tilde{\rho},$$

where

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{r} = \frac{r}{r_0}, \qquad r_0 = \left(\frac{9\sigma^2}{4\pi G\rho_0}\right)^{1/2}, \tag{1}$$

and $\sigma = \left(\frac{kT}{m}\right)^{1/2}$ is the rms 1-d particle velocity. The central boundary conditions are

$$\tilde{\rho}(0) = 1, \qquad \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\tilde{r}} = 0.$$

Write a program that computes the density profile $\tilde{\rho}(\tilde{r})$ of an isothermal sphere out to some specified truncation radius $\tilde{r}_t = r_t/r_0$, where it is assumed to be confined by an external pressure $P_{\rm ext} = P(r_t)$. Note that you can break the second-order differential equation (1) into two first-order differential equations that you integrate simultaneously. Use the midpoint method to obtain second-order accuracy in your integration. In addition to the density itself, have your program compute the scaled mass $M/(r_0^3\rho_0)$ and the scaled values of the potential energy and kinetic thermal energy $W/(GM^2/r_t)$ and $U_{\rm kin}/(GM^2/r_t)$, where

$$W = \int_0^{r_t} \frac{-GM(r)dM}{r}, \qquad U_{\rm kin} = \int_0^{r_t} \frac{3}{2}\frac{P}{\rho}dM.$$

Take enough steps to ensure that each of these quantities converges to a fractional accuracy of 10^{-4}.

(a) Plot the density profile $\tilde{\rho}(\tilde{r})$ out to $\tilde{r} = 30$. Compare your numerical result to the approximate formula $\tilde{\rho}(\tilde{r}) \approx (1+\tilde{r}^2)^{-3/2}$. Over what range is this formula useful?

(b) Give the scaled values of M, W, and $U_{\rm kin}$ for truncated isothermal spheres with $r_t/r_0 = 5$ and $r_t/r_0 = 30$.

(c) For $r_t/r_0 = 5$ and $r_t/r_0 = 30$, compute the value of $P_{\rm ext}$ (choose an appropriate physical scaling). Verify that your numerical solutions satisfy the virial theorem, as you did for the singular isothermal sphere in Part I.