Problem Set 1: The Milne Cosmology
Due Friday, October 5

This special relativistic model, first analyzed by Milne in the 1930s, provides some useful intuition into more realistic cosmological models.

Consider a flat spacetime ($T_{\mu\nu} = 0$ everywhere) in which, at an event BB, particles of zero mass are sprayed in all directions with varying speeds. An observer traveling on one of these particles erects a global coordinate system (t, r, θ, ϕ). In this coordinate system, a particle with speed v (relative to this observer) has reached a distance $r = vt$, so this observer finds that Hubble's law $v = Hr$ holds, with $H = 1/t$.

(1) Suppose that each particle carries a clock, measuring its proper time τ. What are the surfaces $\tau =$ constant in the (t, r, θ, ϕ) coordinate system? Draw one of these surfaces (for fixed θ and ϕ).

(2) Let $r/\tau = \sinh(\omega)$. Argue that (ω, θ, ϕ) are comoving spatial coordinates, i.e. that each particle maintains constant values of ω, θ, and ϕ. For particles with $v \ll 1$, how is ω related to r? Derive the spacetime metric in $(\tau, \omega, \theta, \phi)$ coordinates.

(3) What initial velocity distribution $N(v)$ [number of particles with velocity in the range $v \to v + dv$] is required in order to make the universe homogeneous on surfaces of constant τ?

(4) Now suppose that the particles have non-zero masses. Derive a condition on the mass density such that, if it is satisfied, the gravitational attractions between the particles will not significantly perturb the model. (Conditions like these can usually be cast in the form $|\text{dimensionless number}| \ll 1$.) The expression for this condition may need to involve τ.

(5) Comment on the relation between this cosmology and the Friedmann cosmologies.