Problem Set 4: The First Few Seconds
Due Tuesday, November 13

Introduction

For this problem set, you may find it helpful to refer to Kolb & Turner sections 3.3-3.5 and/or to Peebles pp. 134-139 and 158-165.

For early universe problems, it is often convenient to adopt "high energy physics" units in which $\hbar = c = k_B = 1$ ($k_B =$ Boltzmann's constant) and the fundamental dimension is energy (see Appendix A of Kolb & Turner for discussion). A traditional and convenient unit of energy is $1 \text{ GeV} = 10^9 \text{ eV}$, and in the high energy system of units:

$$1 \text{ GeV} = 1.16 \times 10^{13} \text{ K} = 1.78 \times 10^{-24} \text{ g} = (1.97 \times 10^{-14} \text{ cm})^{-1} = (6.58 \times 10^{-25} \text{ s})^{-1}. \tag{1}$$

Newton's gravitational constant enters into calculations via the Planck mass,

$$m_{Pl} \equiv (\hbar c/G)^{1/2} = G^{-1/2} = 2.18 \times 10^{-5} \text{ g} = 1.22 \times 10^{19} \text{ GeV}. \tag{2}$$

A species of fully relativistic bosons (such as photons) in thermal equilibrium at temperature T has energy density

$$u = \frac{\pi^2}{30} \frac{k_B^4}{(\hbar c)^3} g T^4 = \frac{\pi^2}{30} g T^4, \tag{3}$$

where g is the number of statistical degrees of freedom (e.g., Peebles eq. 6.14). For photons, there are two spin states, so $g = 2$, and the above formula is equivalent to the usual $u = a_B T^4$. For a fully relativistic fermion species (such as neutrinos or relativistic electrons), the result is slightly different because of the different particle statistics:

$$u = \frac{7}{8} \frac{\pi^2}{30} g T^4 \tag{4}$$

(Peebles eq. 6.70). The number density of relativistic particles of a given species and spin state at temperature T is

$$n = \frac{\zeta(3)}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 = 0.122 T^3 \tag{5}$$

for bosons and

$$n = \frac{3\zeta(3)}{4\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 = 0.091 T^3 \tag{6}$$

for fermions (see Peebles eqs. 6.48 and 6.71), where $\zeta(3) \approx 1.202$.

If the energy density of the universe is dominated by relativistic particles and T_γ is the temperature of the photons, equations (3) and (4) together imply that the energy density is

$$\rho = \frac{\pi^2}{30} g_* T_\gamma^4, \tag{7}$$

where

$$g_* \equiv \sum_{i=bosons} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T_\gamma}\right)^4, \tag{8}$$

the sums are over all species of particles relativistic at temperature T_i, and we have allowed for the possibility that each species i is characterized by a different temperature T_i.

When the universe is sufficiently hot and dense, neutrinos and anti-neutrinos are in thermal equilibrium with photons and relativistic electrons and anti-electrons. Coupling between the species ensures that they all have the same temperature $T_\nu = T_e = T_\gamma$, so

$$g_* = 2 + \frac{7}{8} \times [2 \times 2 + 3 \times 2] = 10.75. \tag{9}$$

The first 2 represents the $g = 2$ photon spin states, the 2×2 represents the two spin states each of electrons and anti-electrons, and the 3×2 represents the 3 species of neutrinos (electron, muon, tau) and 3 species of anti-neutrinos. In the standard model of particle physics, neutrinos are always left-handed, so they have only one spin state.

(1) Using the Friedmann equation for a $k = 0$, radiation dominated universe, show that the Hubble parameter $H(T_\gamma)$ at the time that the temperature of the photons in the universe is T_γ is

$$H = \left(\frac{\dot{a}}{a}\right) = 1.66 g_*^{1/2} \frac{T_\gamma^2}{m_{Pl}}. \tag{10}$$

Then show that the age of the universe at temperature T_γ is

$$t(T) = \frac{1}{2H} = 0.30 g_*^{-1/2} \frac{m_{Pl}}{T_\gamma^2} = 0.73 \left(\frac{g_*}{10.75}\right)^{-1/2} \left(\frac{T_\gamma}{1 \text{ MeV}}\right)^{-2} \text{ s}, \tag{11}$$

where 1 MeV = 10^{-3} GeV. (Note: demonstrate each of the three steps of eq. 11).

(2) Argue that a relativistic particle species will remain thermally coupled to other species in the expanding universe as long as

$$n\sigma ct \gg 1, \tag{12}$$

but will decouple from other species once $n\sigma ct \ll 1$, where n is the number density of particles with which the species in question can interact, σ is the typical cross-section for reactions that exchange energy between the particle species, c is the speed of light, and t is the age of the universe.

This argument implies that the condition for a relativistic species to decouple from other species is

$$n\sigma ct = \alpha_d, \tag{13}$$

where $\alpha_d \approx 1$ is a parameter that we can keep track of to understand the sensitivity of further results to the approximate nature of this argument.

(3) For the reactions that couple neutrinos to other species in the early universe, predominantly $\nu + \bar{\nu} \to e^- + e^+$, the typical cross section is $\sigma \approx G_F^2 T^2$ where $G_F = (292.8 \text{ GeV})^{-2}$ is the Fermi coupling constant, which characterizes weak interactions. Using this cross section, your results from (1) and (2), and the number density of neutrinos from equation (6), show that neutrinos decouple when the temperature T is a few MeV.

How sensitive is your result to α_d? Is the assumption that the relativistic species present at neutrino decoupling are neutrinos, photons, and electrons/positrons justified? What is the age of the universe (in seconds) at neutrino decoupling?

(4) What happens to the electrons and positrons when the temperature falls below 0.5 MeV? Using the Thomson cross section for electron-photon interaction, $\sigma = 6.65 \times 10^{-25}$ cm^2, verify that the electrons and positrons are tightly coupled to the photons up to this time.

(5) The annihilating electrons and positrons produce photons that thermalize, adding energy to the photon background. As a result, the temperature of the photons falls slower than $T \propto 1/a$ while electron-positron annihilation is taking place. The neutrinos, on the other hand, are decoupled, so their temperature does fall as $T \propto 1/a$. Therefore, the ratio T_ν/T_γ changes from 1.0 at the time that neutrinos decouple to some value less than 1.0 at times long after electron-positron annihilation.

This eventual ratio T_ν/T_γ can be computed by an entropy argument. Consider a volume V_1 at a time t_1 after neutrino decoupling but before electron-positron annihilation. The entropy of photons and electrons/positrons in this volume is

$$S_1 = S_{\gamma,1} + S_{e,1} = V_1 \times \frac{4}{3} a_B T_{\gamma,1}^3 + V_1 \times 2 \times \frac{7}{8} \times \frac{4}{3} a_B T_{e,1}^3, \tag{14}$$

where a_B is the Stefan-Boltzmann radiation constant (see the discussion in Peebles, near equations 6.51 and 6.74). When electrons and positrons annihilate, their entropy cannot disappear, so it is instead transferred to the photons. The comoving volume $V_2 = V_1(a_2/a_1)^3$ that corresponds to V_1 at some time long after annihilation when the photon temperature is $T_{\gamma,2}$ must therefore have entropy $S_2 = S_1$, but now it resides entirely in the photons.

Build on this argument to show that after electron-positron annihilation, the ratio of the neutrino temperature to the photon temperature is

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}.$$

What is the value of g_* (eq. 8) long after electron-positron annihilation?

(6) The temperature of the cosmic microwave background today is 2.73 K. What is the number density of photons, in cm^{-3}?

Assuming that there are three species of massless neutrinos, what is the number density of neutrinos in the cosmic neutrino background, in cm^{-3}? What is the number density of anti-neutrinos?

Suppose that the heaviest neutrino species, presumably the tau neutrino, has a small but non-zero rest mass of 10 eV, and that the other two species are much lighter. Were the tau neutrinos relativistic or non-relativistic at the time of neutrino decoupling? Are they relativistic or non-relativistic today? What would the contribution of tau neutrinos and anti-neutrinos be to Ω, and should this be considered as a contribution to Ω_r or to Ω_m? (Assume $H_0 = 70$ km s^{-1} Mpc^{-1}.)