A8873: Introduction to Cosmology

Metrics and Metric Notation: Examples

A metric is a matrix (more precisely a tensor) of functions that tells how to go from differential coordinate separations to differential distances (which may be spatial distances or spacetime intervals).

First let's consider two-dimensional space metrics. The spatial separation of nearby points can be written

$$(dl)^2 = \sum_{i,j=1,2} g_{ij} dx^i dx^j = g_{ij} dx^i dx^j \ .$$

I am using standard differential geometry notation with vector indices written as superscripts rather than subscripts. Be careful not to confuse these superscripts with powers. The second equality adopts the Einstein summation convention, in which repeated indices are automatically summed over, which saves a lot of $\sum$ signs but means that a seemingly simple equation can have a lot going on.

Start with Cartesian coordinates on a flat plane, and identify $x^1 = x$, $x^2 = y$. In this case the metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

yielding

$$(dl)^2 = 1 \cdot dx^1 dx^1 + 0 \cdot dx^1 dx^2 + 0 \cdot dx^2 dx^1 + 1 \cdot dx^2 dx^2 = (dx)^2 + (dy)^2 \ .$$

For polar coordinates on a flat plane we identify $x^1 = r$, $x^2 = \theta$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

yielding

$$(dl)^2 = 1 \cdot dx^1 dx^1 + r^2 \cdot dx^2 dx^2 = (dr)^2 + r^2 (d\theta)^2 \ .$$

These examples illustrate two important points. First, even if the underlying space is the same, the metric depends on the coordinate system used to describe it. Second, the metric is in general a function of position (e.g., depending on r in the second case). A constant metric is a special case.

Both of these examples have diagonal metrics. However, if the axes were not orthogonal, then the metric would have off-diagonal terms. The metric is always a symmetric matrix because $dx^i dx^j = dx^j dx^i$.

For polar coordinates on a 2-d spherical surface of curvature radius R, the metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin^2\left(\frac{r}{R}\right) \end{pmatrix} \ .$$

Now consider flat four-dimensional spacetime, with Cartesian spatial coordinates:

$$x^0 = t, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z \ .$$

By convention, the coordinate subscripts for 4-d spacetime are denoted with Greek rather than Latin subscripts.

We are now interested in the flat-spacetime interval separation

$$(ds)^2 = \sum_{\mu,\nu=0,3} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \ .$$

In these coordinates, the metric is

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

yielding

$$\begin{aligned}(ds)^2 &= -c^2 \cdot dx^0 dx^0 + 1 \cdot dx^1 dx^1 + 1 \cdot dx^2 dx^2 + 1 \cdot dx^3 dx^3 \\ &= -c^2 (dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \ .\end{aligned}$$

This is called the Minkowski metric, which can also be written in spherical coordinates.

Relativists frequently work in units where $c = 1$ by definition.

In these units the metric for flat spacetime in Cartesian coordinates is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$