IV. Homogeneous Cosmological Models: Dynamics

The Friedmann Equations

Gravity hasn't entered the picture yet. But to go any further, we need $a(t)$.

Assume GR is correct. We could get equations for $a(t)$ by plugging the FRW metric into the field equation. This yields two non-trivial equations, one of which is the integral of the other.

Instead of following this derivation, we'll use the Newtonian limit, $\nabla^2\Phi = 4\pi G(\rho + 3p)$, which will get us almost all the way. (Caution, I'm still using $c = 1$ here so that ρ and p have the same units.)

My order of doing things is different from Huterer's, though the end results are the same.

We appeal to Birkhoff's theorem, which implies that we can think about a small spherical volume in isolation, ignoring the gravitational effects of the rest of the universe (which cancel out in spherical symmetry).

We will typically think of ρ as a mass density, but for more general forms of energy we could substitute $\rho = \epsilon/c^2$ where ϵ is the energy density.

Consider a shell of physical radius R comoving with the Hubble flow:

$$\ddot{R} = -\frac{4\pi}{3}G(\rho + 3p)R^3 \times \frac{1}{R^2}.$$

But $R = ar$ with r constant, so $\ddot{R} = \ddot{a}r$. Thus,

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a = -\frac{4\pi}{3}G\left[3(\rho + p)a - 2\rho a\right].$$

This equation can be written

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right), \tag{4.1}$$

where I put in the c^2 for clarity [Huterer 3.7].

If we write this in terms of energy density (like Ryden does):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\epsilon + 3p\right).$$

This is an "acceleration" equation for the cosmic expansion. We see already that $\ddot{a} < 0$ if $\rho + 3p > 0$, gravity slows expansion.

We would like to have an "energy" equation for $\dot{a}$, which we can get by integrating if we know how ρ and p change with a.

Use the first law of thermodynamics (energy conservation), assuming that the expansion is adiabatic:

$$-pdV = dU = d(\rho V) = \rho dV + V d\rho$$
$$\Longrightarrow d\rho = -(\rho + p)\frac{dV}{V} = -3(\rho + p)\frac{da}{a}$$

Differentiate w.r.t. t and rearrange to get

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0, \tag{4.2}$$

where I again inserted the correct power of c^2 [Huterer 3.6].

In energy density this would be written

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + p) = 0.$$

This is the continuity equation for an adiabatic fluid in an expanding universe. The adiabatic assumption — no change of entropy — is valid during most of the cosmic expansion, but it is violated at some special epochs when the number of particles changes substantially.

For our immediate purposes it is useful to instead write

$$\dot{a} = \frac{-a}{3(\rho + p)}\dot{\rho}.$$

Multiply both sides of the acceleration equation by $\dot{a}$ to get

$$\dot{a}\ddot{a} = -\frac{4\pi}{3}G\left[-a^2\dot{\rho} - 2\rho a\dot{a}\right]$$
$$= \frac{4\pi}{3}G\left[a^2\dot{\rho} + \rho\frac{d(a^2)}{d\tau}\right].$$

Recognize that $\dot{a}\ddot{a} = d(\dot{a}^2/2)/dt$ and that the term in [ ] is $d(a^2\rho)/dt$. Integrate with respect to t to get

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = \text{constant}.$$

Unfortunately, deriving the integration constant really does require the GR field equation.

We can guess that if the density ρ is high, space will be positively curved, and the universe will be gravitationally bound, making the constant (which plays the role of a potential energy) negative.

Conversely, if ρ is low, space will be negatively curved, and the universe will be unbound, with a positive constant.

GR leads to the conclusion that the integration constant is $-kc^2/R_0^2$.

In dimensionally correct form, the Friedmann equation can be written

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2 R_0^2} = 0. \tag{4.3}$$

[Huterer 2.46] There is no p in this equation, but if we have forms of energy other than matter we should substitute $\rho = \epsilon/c^2$ where ϵ is the energy density.

We will sometimes refer to the first term as the "kinetic" term, the second as the "gravitational" term, and the third as the "curvature" term.

The density parameter

Note that $\dot{a}/a = H$, so if $k = 0$ the Friedmann equation $\Longrightarrow \rho = 3H^2/(8\pi G)$. Define the "critical density"

$$\rho_c \;=\; \frac{3H^2}{8\pi G} \;=\; \text{density of a } k = 0 \text{ Friedmann universe.} \tag{4.3}$$

We can define a dimensionless "cosmological density parameter"

$$\Omega = \frac{\rho}{\rho_c} \tag{4.4}$$

If $\Omega \ll 1$ then the gravitational term of the Friedmann equation is much smaller in magnitude than the kinetic and curvature terms.

The Friedmann equation can also be written

$$H^2(1-\Omega) = \frac{-kc^2}{a^2 R_0^2}.$$

Matching signs implies

$$\begin{aligned} \Omega > 1 &\longrightarrow k = +1, \text{ closed universe} \\ \Omega = 1 &\longrightarrow k = 0, \text{ flat universe} \\ \Omega < 1 &\longrightarrow k = -1, \text{ open universe} \end{aligned}$$

Note that $\Omega = \Omega(t)$, but because k doesn't change, Ω always remains within whichever of these 3 regimes it starts in.

If we define

$$\Omega_k = 1 - \Omega_{\rm tot} \tag{4.5}$$

where $\Omega_{\rm tot}$ is the sum of all other energy densities relative to ρ_c, then the above equation implies

$$R_0 = \frac{c}{H_0}|\Omega_k|^{-1/2}.$$

Evolution of energy density

Consider an energy component with equation of state $p = w\rho c^2$.

The continuity equation derived previously from the first law of thermodynamics,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0,$$

can be written

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}\ .$$

Integrating yields

$$\rho \propto a^{-3(1+w)}.$$

Pressureless matter: $w = 0$, $\rho \propto a^{-3}$ (dilution)

Radiation: $w = 1/3$, $\rho \propto a^{-4}$ (dilution plus redshift)

A cosmological constant has $\rho =$const. by definition, implying $w = -1$.

If these are the energy components in the universe, then the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{a_0}{a}\right)^3 + \rho_{r,0}\left(\frac{a_0}{a}\right)^4 + \rho_{\Lambda,0}\right] = -\frac{kc^2}{a^2 R_0^2}.$$

Here the subscript 0 can represent any fiducial time t_0.

If it represents the present day, then $a_0/a = (1+z)$.

Note that even if the curvature term is comparable to the gravitational term today, it will be negligible at sufficiently high redshift because the ρ_m and ρ_r terms grow more rapidly with $(1+z)$.

Thus, flat universe ($k = 0$) solutions are always accurate at high z.

You will show in PS 2 that the Friedmann equation can be written in the form

$$H(z) = H_0\left[\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\right]^{1/2}, \qquad (4.6)$$

a form that is frequently useful for cosmological calculations.

The subscript 0 refers to present-day values of the density parameters, but these are often just written Ω_m, Ω_k, etc., with the z-dependence written explicitly when referring to evolving values, e.g., $\Omega_m(z)$.

Solutions of the Friedmann equation: single component universe

Empty universe: $\rho = 0$, $k = -1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{c^2}{a^2 R_0^2}.$$

Solution $a = ct/R_0$, $R_0 = ct_0$.

Metric and expansion rate of the Milne cosmology.

Flat universe: $k = 0$, $\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} .$$

Solution $a \propto t^{2/[3(1+w)]}$.

Pressureless matter: $w = 0$, $a = a_0(t/t_0)^{2/3}$.

Radiation: $w = 1/3$, $a = a_0(t/t_0)^{1/2}$.

(Our standard notation has $a_0 = 1$.)

It is easy to show that the age of the universe for $a \propto t^\alpha$ is

$$t = \alpha \times \frac{1}{H} .$$

Λ-*dominated flat universe*: $k = 0$, $\rho = \rho_\Lambda$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\Lambda.$$

Since $\dot{a} \propto a$, solution is exponential growth:

$$a = a_0 e^{t/t_H}, \qquad t_H = \left(\frac{8\pi G\rho_\Lambda}{3}\right)^{-1/2} .$$

Solutions of the Friedmann equation: two component universe

With two components, you can usually figure out the early and late time behavior from the Friedmann equation, with this form being particularly useful:

$$H^2(z) = H_0^2 \left[\Omega_{\Lambda,0} + (1 - \Omega_0)a^{-2} + \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}\right]$$

where I wrote $1 - \Omega_0$ in place of $\Omega_{k,0}$ and I have expressed the expansion in terms of a (with $a_0 = 1$) instead of $(1 + z)$ to help think about the future when $a > 1$.

Matter + curvature

$$H^2 = H_0^2 \left[\Omega_0 a^{-3} + (1 - \Omega_0)a^{-2}\right]$$

We see that for $\Omega_0 > 1$, H becomes zero at the "turnaround" epoch

$$a_{\rm max} = \frac{\Omega_0}{\Omega_0 - 1}.$$

For $\Omega_0 < 1$, H does not reach zero, so it cannot change sign; an expanding sub-critical universe expands forever.

For $\Omega_0 > 1$, the solution (which can be verified by direct substitution and a bit of algebra) can be written in the parametric form

$$\begin{aligned} a(\theta) &= \frac{a_{\rm max}}{2}(1-\cos\theta), \\ t(\theta) &= \frac{a_{\rm max}}{2}\frac{1}{H_0(\Omega_0-1)^{1/2}}(\theta - \sin\theta) \;=\; \frac{R_0}{c}\times\frac{a_{\rm max}}{2}(\theta - \sin\theta)\ . \end{aligned}$$

Maximum expansion is reached at $\theta = \pi$, and the universe collapses in a "big crunch" at $\theta = 2\pi$.

For $\Omega_0 < 1$, define

$$a_* = \frac{\Omega_0}{1-\Omega_0},$$

and the parametric solution is

$$\begin{aligned} a(\eta) &= \frac{a_*}{2}(\cosh\eta - 1), \\ t(\eta) &= \frac{a_*}{2}\frac{1}{H_0(1-\Omega_0)^{1/2}}(\sinh\eta - \eta) \;=\; \frac{R_0}{c}\times\frac{a_*}{2}(\sinh\eta - \eta)\ . \end{aligned}$$

where η runs from zero to infinity.

At late times ($\eta \gg 1$), the solution approaches $a \propto t$ as universe enters "free expansion."

At early times ($\eta \ll 1$, $\theta \ll 1$), both solutions approach $a \propto t^{2/3}$, as for $k = 0$.

Matter + Λ

For a flat universe with a cosmological constant, $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, and the Friedmann equation can be written

$$\frac{H^2}{H_0^2} = \Omega_{m,0}a^{-3} + (1-\Omega_{m,0}).$$

For $\Omega_{m,0} < 1$, $\Omega_{\Lambda,0} > 0$, and the matter density and cosmological constant are equal at an expansion factor

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}.$$

The relation between time and expansion factor can be written in the cumbersome but explicit form

$$H_0 t = \frac{2}{3(1-\Omega_{m,0})^{1/2}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \left(1+\left(\frac{a}{a_{m\Lambda}}\right)^3\right)^{1/2}\right].$$

At early times

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3},$$

like a flat, matter-dominated universe, while at late times

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{\Omega_{\Lambda,0}} H_0 t),$$

giving the exponentially expanding solution for a Λ-dominated universe.

Curvature, Destiny, Topology

As the above solution shows, a matter dominated $k = +1$ universe eventually collapses, while a matter dominated $k = 0$ or $k = -1$ universe expands forever.

This correspondence of closed geometry with a bound universe and flat/open geometry with an unbound universe continues to hold if radiation is added.

But vacuum energy can change the picture.

We have so far assumed that vacuum energy is a cosmological constant, with energy density that does not change as the universe expands.

More generally we could replace $\Omega_{\Lambda,0}$ in the Friedmann equation with $\Omega_{\phi,0}[\rho_\phi(a)/\rho_{\phi,0}]$ to allow for a general dependence.

For a constant w, $\rho_\phi(a)/\rho_{\phi,0} = a^{-3(1+w)}$.

For the universe to recollapse, we must have $H(a) = 0$ at some time in the future ($a > 1$).

If there is no vacuum energy, $\Omega_{\phi,0} = 0$, then this

must happen if $\Omega_k < 0$

cannot happen if $\Omega_k > 0$.

For $\Omega_{\phi,0} > 0$, recollapse can be avoided if $\rho_\phi(a)/\rho_{\phi,0}$ falls slower than a^{-2}.

Best guess current parameters are $\Omega_{\phi,0} \sim 0.7$, $|\Omega_{k,0}| \ll 1$, $\rho_\phi(a) \sim$ const., implying that the universe could be open, flat, or closed, but that expansion forever is likely.

Future recollapse is possible if $\Omega_{k,0} < 0$ and vacuum energy changes its equation of state and starts to fall faster than a^{-2} in the future.

If $\Omega_{\phi,0} < 0$ (a negative vacuum energy is not favored by observations, but it is not obviously impossible in principle), then one could have an $\Omega_k > 1$ (open) universe that recollapses.

GR does not prohibit the universe from having a complex topology, e.g. a toroidal topology in which heading off in one direction eventually brings you back to where you started.

Thus, in principle, the universe could be negatively curved or flat and still be spatially finite.

There have been some (unconvincing) claims for periodic redshifts that could be interpreted as evidence for complex topology.

People are seriously searching for signs of complex topology in the pattern of CMB anisotropies.