VIII. Linear Fluctuations

This topic is covered in §§9.1-9.4 of Huterer.

Linear perturbation theory in general

Start with an "unperturbed" solution for quantities x_i, $i = 1, 2, ...$

Write equations for $\tilde{x}_i = x_i + \delta x_i$.

Expand, keeping terms linear in δx_i, and subtracting off unperturbed solution to get equations for δx_i.

Hope you can solve them.

In GR, the quantities x_i might be metric coefficients, densities, velocities, etc. GR perturbation theory can be tricky for two reasons.

(1) The equations are very complicated. This is a problem of practice, not of principle.

(2) In a perturbed universe, the "natural" coordinate choice is no longer obvious. Different coordinate choices have different equations. Results for physically measurable quantities should be the same, but the descriptions can look quite different.

For scales $l \ll cH^{-1}$ and velocities $v \ll c$, one can usually get by with Newtonian linear perturbation theory.

Newtonian perturbation theory in an expanding universe

Basic unperturbed equations

In an inertial frame, the equations governing the density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{u}(\mathbf{r}, t)$ of an ideal fluid are:

$$\text{Continuity equation}: \quad \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{r}} + \vec{\nabla}_{\mathbf{r}} \cdot (\rho \mathbf{u}) = 0 \quad \text{(mass conservation)}$$

$$\text{Euler equation}: \quad \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{r}} + (\mathbf{u} \cdot \vec{\nabla}_{\mathbf{r}})\mathbf{u} = -\vec{\nabla}_{\mathbf{r}} \Phi \quad \text{(momentum conservation)}$$

$$\text{Poisson equation}: \quad \nabla_{\mathbf{r}}^2 \Phi = 4\pi G \rho.$$

We have ignored pressure gradients in the Euler equation.

Transformation of variables

We would like to have these equations in comoving coordinates $\mathbf{x} = \mathbf{r}/a(t)$ with "peculiar" velocity $\mathbf{v} = \mathbf{u} - (\dot{a}/a)\mathbf{r} = \dot{(a\mathbf{x})} - \dot{a}\mathbf{x} = a\dot{\mathbf{x}}$.

We want to use a quantity that will be small when perturbations are small, so define the dimensionless density contrast $\delta(\mathbf{x}, t)$ by $\rho = \rho_b(t)\,[1 + \delta(\mathbf{x}, t)]$, ρ_b = background density $\propto 1/a^3$.

By the chain rule, the time derivative of a function f at fixed time t and comoving position $\mathbf{x} = \mathbf{r}/a$ is

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{\mathbf{x}} \cdot \vec{\nabla}_{\mathbf{r}} f = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + \frac{\dot{a}}{a}\frac{\mathbf{r}}{a} \cdot \left(a\vec{\nabla}_{\mathbf{r}} f\right).$$

Rearranging terms yields the expression for the time derivative of f at fixed t and $\mathbf{r}$:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} - \frac{\dot{a}}{a}\mathbf{x} \cdot \vec{\nabla} f, \quad \vec{\nabla} \equiv a\vec{\nabla}_{\mathbf{r}}.$$

With appropriate substitutions, the three equations above become:

$$\begin{aligned} \text{Continuity}: &\quad \frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot [(1+\delta)\mathbf{v}] = 0 \\ \text{Euler}: &\quad \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\mathbf{v} \cdot \vec{\nabla})\mathbf{v} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{1}{a}\vec{\nabla}\phi \\ \text{Poisson}: &\quad \nabla^2\phi = 4\pi G\rho_b a^2 \delta \quad \text{with} \quad \phi \equiv \Phi - \frac{2}{3}\pi G \rho_b a^2 x^2. \end{aligned}$$

The one qualitatively new feature is the "friction" term $\left(\frac{\dot{a}}{a}\right)\mathbf{v}$ in the Euler equation. It drags $\mathbf{v}$ to 0 if $\vec{\nabla}\phi$ vanishes.

This is just the "kinematic redshift" that we encountered long ago; it reflects the use of expanding, non-inertial coordinates.

It will have a crucial consequence: density contrasts in an expanding universe grow as power laws of time (for $\Omega_m = 1$) instead of exponentials.

The linear perturbation equation

So far, the equations are general, within the Newtonian limit and the assumption that pressure gradients are negligible, $\vec{\nabla}p/\rho \ll \vec{\nabla}\phi$.

The linear approximation is obtained by keeping only the terms that are first order in δ or $\mathbf{v}$:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot \mathbf{v} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\vec{\nabla}\phi = 0.$$

Eliminate $\mathbf{v}$ by taking time derivative of first equation, $(1/a) \times$ divergence of second, subtracting, and substituting using the continuity and Poisson equations.

Details:

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{1}{a}\frac{\partial}{\partial t}[\vec{\nabla} \cdot \mathbf{v}] - \frac{\dot{a}}{a^2}\vec{\nabla} \cdot \mathbf{v} = 0$$

and

$$\frac{1}{a}\frac{\partial}{\partial t}[\vec{\nabla} \cdot \mathbf{v}] + \frac{\dot{a}}{a^2}\vec{\nabla} \cdot \mathbf{v} + \frac{1}{a^2}\nabla^2\phi = 0.$$

Subtract:

$$\frac{\partial^2\delta}{\partial t^2} - 2\frac{\dot{a}}{a^2}\vec{\nabla}\cdot\mathbf{v} - \frac{1}{a^2}\nabla^2\phi = 0 \ .$$

From continuity substitute:

$$\frac{1}{a}\vec{\nabla}\cdot\mathbf{v} = -\frac{\partial\delta}{\partial t} \ .$$

Substitute from Poisson equation for $\nabla^2\phi$.

Result:

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} = 4\pi G\rho_b\delta.$$

This is a second-order differential equation for $\delta(\mathbf{x},t)$ with a growing and decaying mode solution:

$$\delta(\mathbf{x},t) = A(\mathbf{x})D_1(t) + B(\mathbf{x})D_2(t).$$

The linear growth factor

We can rewrite the equation for δ as an equation for the growth factor

$$\ddot{D} + 2H(z)\dot{D} - \frac{3}{2}\Omega_{m,0}H_0^2(1+z)^3D = 0. \tag{8.1}$$

For $\Omega_m = 1$, $D_1(t) \propto t^{2/3} \propto a(t)$, $D_2(t) \propto t^{-1} \propto a^{-3/2}$.

A more complicated algebraic expression for $D_1(t)$ in the case of $\Omega_m < 1$, $\Omega_\Lambda = 0$ is given in Peebles (1980, Large Scale Structure of the Universe), equation 11.16.

For a flat universe with a cosmological constant, the solution to the differential equation can be written in integral form

$$D_1(z) = \frac{H(z)}{H_0}\int_z^\infty \frac{dz'(1+z')}{H^3(z')}\left[\int_0^\infty \frac{dz'(1+z')}{H^3(z')}\right]^{-1} ,$$

where the factor in brackets makes the normalization $D_1 = 1$ at $z = 0$.

For other equations of state of dark energy, or for more components (curvature, radiation), no general integral expression exists, and the differential equation must be solved directly.

To a good approximation, $d\ln D_1/d\ln a \approx \Omega_m^{0.55}$, implying $D_1 \propto a$ while $\Omega_m \approx 1$ and slowing of growth as Ω_m falls below one.

The linear density and velocity fields

If we assume that density fluctuations appear at some very early time, then at a much later time we care only about the growing mode $D_1(t)$, which we can just call $D(t)$. Thus,

$$\delta(\mathbf{x},t) = \delta(\mathbf{x},t_i)\frac{D(t)}{D(t_i)}, \qquad D(t) \propto a(t) \propto t^{2/3} \text{ for } \Omega_m = 1. \tag{8.2}$$

As long as $\delta \ll 1$, density contrasts simply "grow in place" in comoving coordinates.

From the continuity equation

$$\vec{\nabla} \cdot \mathbf{v} = -a \frac{\partial \delta}{\partial t} = -a\delta \frac{\dot{D}}{D} = -a\delta H f(\Omega_m), \tag{8.3}$$

where

$$f(\Omega_m) \equiv \frac{1}{H} \frac{\dot{D}}{D} = \frac{d\mathrm{ln}D}{d\mathrm{lna}} \approx \Omega_m^{0.55}. \tag{8.4}$$

The condition for a perturbation to be growing mode is $\dot{\delta} = \delta H f(\Omega_m)$; growing mode perturbations are those for which the velocity divergence correctly reinforces the gravitational growth.

Newtonian linear perturbation theory, bottom lines

- For $\Omega_m = 1$, $\delta(\mathbf{x}, t) \propto a(t) \propto t^{2/3}$ [growing mode, matter dominated, no pressure].
- For $\Omega_m \neq 1$, $\delta(\mathbf{x}, t) = \delta(\mathbf{x}, t_i) D(t)/D(t_i)$, but $D(t)$ is more complicated. Generally $d \ln D / d \ln a = f(\Omega_m) \approx \Omega_m^{0.55}$, so fluctuation growth slows when Ω_m drops below 1.
- The peculiar velocity field satisfies $\vec{\nabla} \cdot \mathbf{v} = -a\delta H f(\Omega_m)$.
- Pressure gradients can stabilize fluctuations on scales smaller than $c_s (G\rho_b)^{-1/2}$, where c_s is the sound speed.
- Because of kinematic redshift, linear perturbations grow roughly as power laws of time rather than exponentially, so they do not forget their initial conditions.
- All this changes when $|\delta|$ reaches ~ 1.

Fourier description

It is often convenient to work with Fourier components of δ:

$$\delta(\mathbf{x}, t) = \int d^3k \delta_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}, \qquad \delta_{\mathbf{k}} = \int d^3x \delta(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}.$$

Each $\delta_{\mathbf{k}} = A_{\mathbf{k}} e^{i\theta_{\mathbf{k}}}$ is a complex number, the Fourier amplitude. (Beware that different authors choose different conventions on where to put the 2π's.)

If the phases $\theta_{\mathbf{k}}$ are uncorrelated, then the field $\delta(\mathbf{x})$ is *Gaussian.* This implies that the 1-point probability distribution function of δ is Gaussian,

$$P(\delta) = (2\pi\sigma^2)^{-1/2} e^{-\delta^2/2\sigma^2},$$

where the variance

$$\sigma^2 \equiv \langle \delta^2 \rangle = \int_0^\infty 4\pi k^2 P(k) dk = \int_0^\infty 4\pi k^3 P(k) d\ln k\ . \tag{8.5}$$

Here $P(k)$ is the *power spectrum*

$$P(k) \equiv \langle A_{\mathbf{k}} A_{\mathbf{k}}^* \rangle, \tag{8.6}$$

which tells the variance as a function of scale (whether or not the field is Gaussian). The second equality shows that $4\pi k^3 P(k)$ is the contribution to the variance from a (natural) logarithmic range of k.

Note that $P(k)$ has units of volume and $k^3 P(k)$ is dimensionless.

A commonly used Fourier convention, without the 2π's in the exponent, instead gives

$$\sigma^2 \equiv \langle \delta^2 \rangle = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} d\ln k$$

(Huterer eq. 9.60).

In linear theory, each Fourier mode evolves independently, $\delta_{\mathbf{k}}(t) \propto D(t)$.

The spatial scale is $\lambda = 2\pi/k$, so even if small scale modes have become nonlinear, $4\pi k^3 \delta_{\mathbf{k}} \gtrsim 1$, large scale modes may still follow linear theory. (This statement is not obviously true, but it holds in most circumstances.)

On scales in the linear regime, the shape of the power spectrum is preserved, and its amplitude grows $\propto D^2(t)$.

Variance of mass fluctuations

We are often interested in the properties of the density field after it has been smoothed, by convolution with a window function $W_R(r)$ of characteristic radius R.

Common choices for $W(r)$ include a top hat,

$$W(r) = \Theta\left[1 - \frac{r}{R}\right]$$

and a Gaussian

$$W(r) = e^{-r^2/2R^2}.$$

The variance of fluctuations of the smoothed density field is

$$\begin{aligned}\sigma^2(R) &= \int_0^\infty 4\pi k^2 P(k) dk\, \widetilde{W}^2(kR) \\ &= \int_0^\infty \Delta^2(k) d\ln k\, \widetilde{W}^2(kR),\end{aligned}$$

where $\Delta^2(k) \equiv 4\pi k^3 P(k)$

and

$$\widetilde{W}(kR) = \int d^3 r W_R(r) e^{-2\pi i \mathbf{k}\cdot\mathbf{r}}$$

is the Fourier transform of the window function.

If $W_R(r)$ depends only on r/R, then $\widetilde{W}(\mathbf{k})$ depends on $\mathbf{k}$ only through the combination kR, and it generally satisfies

$$\widetilde{W} \approx 1 \qquad 2\pi kR \ll 1$$
$$\widetilde{W} \approx 0 \qquad 2\pi kR \gg 1.$$

For a power-law power spectrum

$$P(k) \propto k^n \Longrightarrow \sigma(R) \propto R^{-(3+n)/2} \propto M^{-(3+n)/6}$$

provided $n > -3$. The variance is dominated by waves with $k \sim 1/(2\pi R)$.

$n = 0$ corresponds to a "white noise" power spectrum, for which we have

$$\left(\frac{\delta M}{M}\right)_{rms} \propto M^{-1/2}$$

as we would naively guess.

On scales of a few h^{-1} Mpc, the observed power spectrum slope is $n \sim -1$, so fluctuations decrease more slowly with increasing scale than they would for white noise.

These results hold generally, not just in linear theory, provided that one uses the non-linear $P(k)$ rather than the linear $P(k)$. However, if one starts with Gaussian fluctuations, the probability distribution $P(\delta)$ becomes non-Gaussian in the non-linear regime.

It is conventional to describe the amplitude of the linear matter power spectrum $P(k)$ by σ_8, the rms fluctuation $\sigma(R)$ of matter fluctuations smoothed with a top-hat sphere of radius $R = 8h^{-1}$ Mpc.

This convention arose because the rms fluctuation of luminous (roughly L_*) galaxies is about one at $8h^{-1}$ Mpc.

Note that σ_8 conventionally refers to a calculation using the *linear* $P(k)$ even though it's a scale that is mildly non-linear. Unless otherwise specified, σ_8 usually refers to fluctuations linearly extrapolated to $z = 0$, and at higher redshifts one writes $\sigma_8(z)$.