Problem Set 2: Cosmological Parameters and Their Evolution

Due Wednesday, February 14

You should allocate 6-8 hours to this problem set and stop if you hit that limit. Please upload to Carmen, as that proves the easiest way for me to grade.

Recall that

$$\begin{aligned} \text{the redshift} \quad & z \equiv \frac{a_0}{a} - 1 \\ \text{the Hubble parameter} \quad & H \equiv \frac{\dot{a}}{a} \\ \text{the critical density} \quad & \rho_c \equiv \frac{3H^2}{8\pi G}\ , \\ \text{the density parameter} \quad & \Omega \equiv \frac{\rho}{\rho_c}\ . \end{aligned}$$

where a_0 is the value of the expansion factor at $z = 0$. In our standard notation (adopted below), $a_0 \equiv 1$, so $z = a^{-1} - 1$.

The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2 R_0^2} = 0, \qquad k = -1,\ 0,\ \text{or}\ +1,$$

where ρ is the *total* energy density.

(1) Show that the Friedmann equation can be written

$$H^2(1-\Omega) = -\frac{kc^2}{a^2R_0^2}\ .$$

What is the relation between the values of Ω and k? For $k \neq 0$, what is the value of the curvature radius $R(t) = a(t)R_0$ in terms of H and Ω?

(2) Now assume that the energy components of the universe are matter, radiation, and a constant vacuum energy (i.e., a cosmological constant). The energy densities of these components scale with redshift as

$$\begin{aligned} \rho_m &= \rho_{m,0}(1+z)^3 \\ \rho_r &= \rho_{r,0}(1+z)^4 \\ \rho_\Lambda &= \rho_{\Lambda,0} = \text{const.}, \end{aligned}$$

where subscripts zero denote values at $z = 0$. Define

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$$

implying

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda.$$

Further define

$$\Omega_k = -\frac{kc^2}{a^2R_0^2H^2}.$$

[Note that $\Omega_k = -k(D_H/R)^2$ where $D_H(z) \equiv cH^{-1}(z)$ is the Hubble distance.]

Use the Friedmann equation to show that

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1.$$

(3) Show that the Friedmann equation can be written in the form

$$H(z) = H_0 \left[\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\right]^{1/2}, \tag{1}$$

where subscripts 0 refer to the present day values of parameters. (Hint: Draw on your solution from part (1) and think carefully about how ρ and ρ_c change with redshift.)

(4) Show that the lookback time to an object at redshift z (i.e., the elapsed time from redshift z to redshift 0) is

$$t = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z') \left[\Omega_{\Lambda,0} + \Omega_{k,0}(1+z')^2 + \Omega_{m,0}(1+z')^3 + \Omega_{r,0}(1+z')^4\right]^{1/2}} \ . \tag{2}$$

Hint: convert dt to da, then da to dz.

(5) Show that

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4}. \tag{3}$$

(6) Recall that the comoving distance is

$$r = D_c = \frac{c}{H_0} \int_0^z dz' \frac{H_0}{H(z')} \ . \tag{4}$$

Set $\Omega_{r,0} = \Omega_{k,0} = 0$. Write a computer program that evaluates the integrals in equation (2) and equation (4) to compute age and comoving distance as a function of redshift for a flat universe with user-specified H_0 and $\Omega_{m,0}$, with $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

Make plots of $\Omega_m(z)$ (from equation 3), $t(z)$, and $D_c(z)$ over the range $z = 0 - 6$ for $H_0 = 70$ km/s/Mpc for flat universe models with $\Omega_{m,0} = 1$ and $\Omega_{m,0} = 0.3$.

I have provided a python program via Carmen that you can use as a starting point if you wish.

Extended goal if time permits

Adjust your program to allow for a non-flat universe, so that $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are both specified by the user. Add curves for an open universe with $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0$ to your previous plots.

Using the formulas provided in the notes, have your program compute the angular diameter distance $D_A(z)$ in addition to $D_C(z)$.

Make a plot comparing $D_A(z)$ for a flat universe with $\Omega_m = 1$, a flat universe with $\Omega_{m,0} = 1 - \Omega_{\Lambda,0} = 0.3$, and an open universe with $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0$.